Supplementary material #1 for manuscript "Time-frequency analysis of event-related brain recordings: Connecting power of evoked potential and inter-trial coherence"

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Contents

1 Expression of avgAMP, ITC, and POWavg

1.1 avgAMP

The square of avgAMP can be expanded as

$$\begin{aligned}\text{avgAMP}_{x_{1:N}}(t,f)^2 &= \left[\frac{1}{N}\sum_{n=1}^{N}|T_{x_n}(t,f)|\right]^2 \\ &= \frac{1}{N^2}\left[\sum_{n=1}^{N}|T_{x_n}(t,f)|^2 + \sum_{m\neq n}|T_{x_m}(t,f)|\,|T_{x_n}(t,f)|\right]. \qquad \text{(S-1)}\end{aligned}$$

Its expectation yields

$$\begin{aligned}\mathrm{E}\left[\text{avgAMP}_{x_{1:N}}(t,f)^2\right] &= \frac{1}{N}\mathrm{E}\left[|T_x(t,f)|^2\right] + \left(1-\frac{1}{N}\right)\mathrm{E}\left[|T_x(t,f)|\right]^2 \\ &= \mathrm{E}\left[|T_x(t,f)|\right]^2 + \frac{1}{N}\mathrm{Var}\left[|T_x(t,f)|\right]. \qquad \text{(S-2)}\end{aligned}$$

1.2 ITC

The square of ITC can be expanded as

$$\text{ITC}_{x_{1:N}}(t,f)^2 = \frac{1}{N} + \frac{1}{N^2}\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}. \qquad \text{(S-3)}$$

Its expectation is given by

$$\begin{aligned}\mathrm{E}\left[\text{ITC}_{x_{1:N}}(t,f)^2\right] &= \frac{1}{N} + \left(1-\frac{1}{N}\right)\left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right|^2 \\ &= \left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right|^2 + \frac{1}{N}\mathrm{Var}\left[e^{i\theta_x(t,f)}\right], \end{aligned}$$

$$\text{(S-4)}$$

where we used the fact that, according to Equation (7) of the manuscript, we have

$$\mathrm{Var}\left[e^{i\theta_x(t,f)}\right] = 1 - \left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right|^2. \qquad \text{(S-5)}$$

1.3 POWavg

We have

$$\begin{aligned}\text{POWavg}_{x_{1:N}}(t,f) &= \left[\frac{1}{N}\sum_{n=1}^{N}T_{x_n}(t,f)\right]\left[\frac{1}{N}\sum_{n=1}^{N}T_{x_n}(t,f)\right]^* \\ &= \frac{1}{N^2}\left[\sum_{n=1}^{N}|T_{x_n}(t,f)|^2 + \sum_{m\neq n}T_{x_m}(t,f)T_{x_n}(t,f)^*\right].\end{aligned}$$

Taking the expectation yields

$$\mathrm{E}\left[\text{POWavg}_{x_{1:N}}(t,f)\right] = \frac{1}{N^2}\left\{\sum_{n=1}^{N}\mathrm{E}\left[|T_{x_n}(t,f)|^2\right] + \sum_{n\neq m}\mathrm{E}\left[T_{x_n}(t,f)T_{x_m}(t,f)^*\right]\right\}.$$

Since the $T_{x_n}(t,f)$'s are i.i.d. realizations of $T_x(t,f)$, we obtain

$$\begin{aligned}\mathrm{E}\left[T_{x_n}(t,f)T_{x_m}(t,f)^*\right] &= \mathrm{E}\left[T_{x_n}(t,f)\right]\mathrm{E}\left[T_{x_m}(t,f)\right]^* \\ &= \left|\mathrm{E}\left[T_x(t,f)\right]\right|^2\end{aligned}$$

and

$$\begin{aligned} \mathrm{E}\left[\mathrm{POWavg}_{x_{1:N}}(t,f)\right] &= \frac{1}{N}\mathrm{E}\left[|T_x(t,f)|^2\right] + \left(1-\frac{1}{N}\right)|\mathrm{E}\left[T_x(t,f)\right]|^2 \\ &= |\mathrm{E}\left[T_x(t,f)\right]|^2 + \frac{1}{N}\mathrm{Var}\left[T_x(t,f)\right]. \end{aligned} \tag{S-6}$$

1.4 Summary of properties

Results regarding the properties of the expectation of avgAMP^2, ITC^2 and POWavg are summarized here.

Measure	Expectation		
	depends on	limit as $N \to \infty$	asymptotic expectation
avgAMP^2	amplitude, $\lvert T_x(t,f)\rvert$	$\mathrm{E}\left[\lvert T_x(t,f)\rvert\right]^2$	$\mathrm{E}\left[\lvert T_x(t,f)\rvert\right]^2 + O\left(\frac{1}{N}\right)$
ITC^2	phase, $\theta_x(t,f)$	$\left\lvert\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right\rvert^2$	$\left\lvert\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right\rvert^2 + O\left(\frac{1}{N}\right)$
POWavg	both, $T_x(t,f)$	$\lvert\mathrm{E}\left[T_x(t,f)\right]\rvert^2$	$\lvert\mathrm{E}\left[T_x(t,f)\right]\rvert^2 + O\left(\frac{1}{N}\right)$

2 Investigation of oscillatory model

2.1 Preliminary results

We need the values of the three following integrals. The first integral is

$$I_1(t,\alpha,f) = \frac{1}{\sqrt{2\pi\alpha^2}}\int e^{-\frac{(u-t)^2}{2\alpha^2}}\, e^{-2i\pi f u}\,\mathrm{d}u. \tag{S-7}$$

It can be obtained as the characteristic function of a normal distribution (Polyanin and Manzhirov, 2007, Equation (20.2.4.6)) computed at $-2\pi f$,

$$I_1(t,\alpha,f) = e^{-\frac{1}{2}(2\pi\alpha f)^2} e^{-2i\pi f t}. \tag{S-8}$$

The second integral is

$$I_2(\Omega,t,\alpha,f) = \frac{1}{\sqrt{2\pi\alpha^2}}\int \Omega\cos(2\pi\nu u+\phi)\, e^{-\frac{(u-t)^2}{2\alpha^2}}\, e^{-2i\pi f u}\,\mathrm{d}u. \tag{S-9}$$

It can be computed by first using Euler formula for the cosine function (Polyanin and Manzhirov, 2007, §2.2.3-14)

$$\begin{aligned} I_2(\Omega,t,\alpha,f) &= \frac{\Omega}{\sqrt{2\pi\alpha^2}}\int \frac{e^{i(2\pi\nu u+\phi)}+e^{-i(2\pi\nu u+\phi)}}{2}\, e^{-\frac{(u-t)^2}{2\alpha^2}}\, e^{-2i\pi f u}\,\mathrm{d}u \\ &= \frac{\Omega}{2}e^{i\phi}\frac{1}{\sqrt{2\pi\alpha^2}}\int e^{-\frac{(u-t)^2}{2\alpha^2}}\, e^{-2i\pi(f-\nu)u}\,\mathrm{d}u + \frac{\Omega}{2}e^{-i\phi}\frac{1}{\sqrt{2\pi\alpha^2}}\int e^{-\frac{(u-t)^2}{2\alpha^2}}\, e^{-2i\pi(f+\nu)u}\,\mathrm{d}u, \end{aligned} \tag{S-10}$$

and then integrating each exponential using (S-8):

$$\begin{aligned} I_2(\Omega,t,\alpha,f) &= \frac{\Omega}{2}e^{i\phi}I_1(t,\alpha,f-\nu) + \frac{\Omega}{2}e^{-i\phi}I_1(t,\alpha,f+\nu) \\ &= \frac{\Omega}{2}e^{-\frac{1}{2}(2\pi\alpha)^2(f-\nu)^2}e^{i[\phi-2\pi(f-\nu)t]} + \frac{\Omega}{2}e^{-\frac{1}{2}(2\pi\alpha)^2(f+\nu)^2}e^{i[-\phi-2\pi(f-\nu)t]}. \end{aligned} \tag{S-11}$$

The third and last integral is

$$I_3(\Omega,t,\alpha,t_0,\tau,f) = \frac{1}{\sqrt{2\pi\alpha^2}}\int \Omega\cos(2\pi\nu u+\phi)\, e^{-\frac{(u-t_0)^2}{2\tau^2}}\, e^{-\frac{(u-t)^2}{2\alpha^2}}\, e^{-2i\pi f u}\,\mathrm{d}u. \tag{S-12}$$

It can be calculated by first reorganizing the quadratic terms of the exponential

$$\begin{aligned} Q &= \frac{1}{\tau^2}(u-t_0)^2 + \frac{1}{\alpha^2}(u-t)^2 \\ &= \left(\frac{1}{\tau^2}+\frac{1}{\alpha^2}\right)u^2 - 2u\left(\frac{t_0}{\tau^2}+\frac{t}{\alpha^2}\right) + \frac{t_0^2}{\tau^2}+\frac{t^2}{\alpha^2}. \end{aligned}$$

Setting

$$\hat{t} = \frac{\frac{t_0}{\tau^2}+\frac{t}{\alpha^2}}{\frac{1}{\tau^2}+\frac{1}{\alpha^2}},$$

we obtain

$$Q = \left(\frac{1}{\tau^2}+\frac{1}{\alpha^2}\right)(u-\hat{t})^2 + \frac{t_0^2}{\tau^2}+\frac{t^2}{\alpha^2} - \left(\frac{1}{\tau^2}+\frac{1}{\alpha^2}\right)\hat{t}^2.$$

The term that does not depend on u can be expanded and simplified to yield

$$\begin{aligned} \frac{t_0^2}{\tau^2}+\frac{t^2}{\alpha^2} - \left(\frac{1}{\tau^2}+\frac{1}{\alpha^2}\right)\hat{t}^2 &= \frac{\frac{1}{\tau^2}\frac{1}{\alpha^2}}{\frac{1}{\tau^2}+\frac{1}{\alpha^2}}(t-t_0)^2 \\ &= \frac{1}{\tau^2+\alpha^2}(t-t_0)^2, \end{aligned}$$

so that

$$Q = \left(\frac{1}{\tau^2}+\frac{1}{\alpha^2}\right)(u-\hat{t})^2 + \frac{(t-t_0)^2}{\tau^2+\alpha^2}$$

and

$$I_3(t,\alpha,t_0,f) = e^{\frac{(t-t_0)^2}{2(\tau^2+\alpha^2)}} \frac{1}{\sqrt{2\pi\alpha^2}} \int \Omega\cos(2\pi\nu u+\phi)\, e^{-\frac{1}{2}\left(\frac{1}{\tau^2}+\frac{1}{\alpha^2}\right)(u-\hat{t})^2}\, e^{-2i\pi f u}\, \mathrm{d}u. \tag{S-13}$$

Setting

$$\beta = \frac{1}{\tau^2}+\frac{1}{\alpha^2}, \tag{S-14}$$

we can then applying (S-10):

$$\begin{aligned} I_3(t,\alpha,t_0,f) &= \frac{\beta}{\alpha} e^{\frac{(t-t_0)^2}{2(\tau^2+\alpha^2)}} \frac{1}{\sqrt{2\pi\beta^2}} \int \Omega\cos(2\pi\nu u+\phi)\, e^{-\frac{(u-\hat{t})^2}{2\beta^2}}\, e^{-2i\pi f u}\, \mathrm{d}u \\ &= \frac{\beta}{\alpha} e^{\frac{(t-t_0)^2}{2(\tau^2+\alpha^2)}} I_2\left(\Omega,\hat{t},\beta,f\right) \\ &= \frac{\beta}{\alpha} e^{-\frac{(t-t_0)^2}{2(\tau^2+\alpha^2)}} \left\{ \frac{\Omega}{2} e^{-\frac{1}{2}(2\pi\beta)^2(f-\nu)^2} e^{i[\phi-2\pi(f-\nu)\hat{t}]} + \frac{\Omega}{2} e^{-\frac{1}{2}(2\pi\beta)^2(f+\nu)^2} e^{i[-\phi-2\pi(f-\nu)\hat{t}]} \right\}. \end{aligned} \tag{S-15}$$

Note that, for $\tau^2 \ll \alpha^2$, we have $\beta^2 \approx \tau^2$, whereas $\beta^2 \approx \alpha^2$ for $\tau^2 \gg \alpha^2$.

2.2 S-transform of oscillatory signal

We here calculate the time-frequency transform of a signal of the form given by Equation (24) of the manuscript. Application of (S-10) with $\alpha = 1/f$ yields

$$\begin{aligned} T_x(t,f) &= \frac{|f|}{\sqrt{2\pi}} \int \Omega\cos(2\pi\nu u+\phi)\, e^{-\frac{f^2(u-t)^2}{2}}\, e^{-2i\pi f u}\, du \\ &= \frac{\Omega}{2} e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu}{f}\right)^2} e^{i[\phi-2\pi(f-\nu)t]} + \frac{\Omega}{2} e^{-\frac{1}{2}(2\pi)^2\left(1+\frac{\nu}{f}\right)^2} e^{i[-\phi-2\pi(f-f_0)t]}. \end{aligned} \tag{S-16}$$

2.3 Approximation

2.3.1 General approach

We here provide an approximation for the modulus and argument of $T_x(t,f)$ in (S-16). We first express $T_x(t,f)$ as

$$T_x(t,f) = \frac{\Omega}{2} e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu}{f}\right)^2} e^{i[\phi-2\pi(f-\nu)t]} \left[1 + e^{-8\pi^2\frac{\nu}{f}} e^{i(-2\phi-4\pi\nu t)}\right].$$

Setting

$$T_x^\dagger(t,f) = \frac{\Omega}{2} e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu}{f}\right)^2} e^{i[\phi-2\pi(f-\nu)t]} \tag{S-17}$$

and

$$\epsilon(t,f) = e^{-8\pi^2\frac{\nu}{f}} e^{i(-2\phi-4\pi\nu t)}, \tag{S-18}$$

we can express $T_x(t,f)$ as

$$T_x(t,f) = T_x^\dagger(t,f)\left[1+\epsilon(t,f)\right]. \tag{S-19}$$

From there, the module and argument of $T_x(t,f)$ can be calculated as

$$|T_x(t,f)| = \left|T_x^\dagger(t,f)\right| |1+\epsilon(t,f)| \tag{S-20}$$

$$\arg\left[T_x(t,f)\right] = \arg\left[T_x^\dagger(t,f)\right] + \arg\left[1+\epsilon(t,f)\right]. \tag{S-21}$$

The module and argument of $T_x^\dagger(t,f)$ can be easily calculated, yielding

$$\left|T_x^\dagger(t,f)\right| = \frac{\Omega}{2} e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu}{f}\right)^2} \tag{S-22}$$

$$\arg\left[T_x^\dagger(t,f)\right] = \phi - 2\pi(f-\nu)t. \tag{S-23}$$

The module and argument of $1+\epsilon(t,f)$ are not quite as straightforward to obtain. See Figure 1 for a schematic description. Since we only consider $f>0$, we have $e^{-8\pi^2\frac{\nu}{f}} < 1$, so that $0 < \epsilon(t,f) < 1$ and $1+\epsilon(t,f)$ has modulus in $]0,2[$ and argument in $]-\frac{\pi}{2},\frac{\pi}{2}[$. We define $\epsilon_m(t,f)$ and $\epsilon_a(t,f)$ as

$$|1+\epsilon(t,f)| = 1+\epsilon_m(t,f) \tag{S-24}$$

$$\arg\left[1+\epsilon(t,f)\right] = \epsilon_a(t,f). \tag{S-25}$$

We now provide bounds for both quantities.

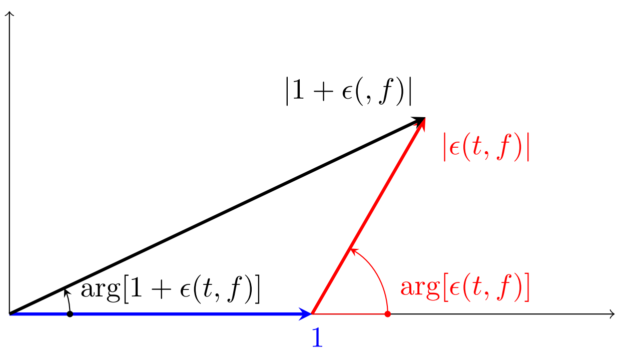

Figure 1: **Modulus and argument of $1+\epsilon(t,f)$.**

2.3.2 Bounds for $\epsilon_m(t,f)$

From (S-24) and (S-18), we have

$$\begin{aligned}
[1+\epsilon_m(t,f)]^2 &= |1+\epsilon(t,f)|^2 \\
&= \left|1+e^{-8\pi^2\frac{\nu}{f}}e^{i(-2\phi-4\pi\nu t)}\right|^2 \\
&= \left[1+e^{-8\pi^2\frac{\nu}{f}}\cos(2\phi+4\pi\nu t)\right]^2 \\
&\quad +\left[e^{-8\pi^2\frac{\nu}{f}}\sin(2\phi+4\pi\nu t)\right]^2 \\
&= 1+2e^{-8\pi^2\frac{\nu}{f}}\cos(2\phi+4\pi\nu t) \\
&\quad +e^{-16\pi^2\frac{\nu}{f}},
\end{aligned}$$

so that

$$\begin{aligned}
[1+\epsilon_m(t,f)]^2 &< 1+2e^{-8\pi^2\frac{\nu}{f}}+e^{-16\pi^2\frac{\nu}{f}} \\
&< \left(1+e^{-8\pi^2\frac{\nu}{f}}\right)^2 \\
1+\epsilon_m(t,f) &< 1+e^{-8\pi^2\frac{\nu}{f}} \qquad \text{(S-26)}
\end{aligned}$$

and

$$\begin{aligned}
[1+\epsilon_m(t,f)]^2 &> 1-2e^{-8\pi^2\frac{\nu}{f}}+e^{-16\pi^2\frac{\nu}{f}} \\
&> \left(1-e^{-8\pi^2\frac{\nu}{f}}\right)^2 \\
1+\epsilon_m(t,f) &> 1-e^{-8\pi^2\frac{\nu}{f}}. \qquad \text{(S-27)}
\end{aligned}$$

From (S-26) and (S-27), we are led to

$$|\epsilon_m(t,f)| < e^{-8\pi^2\frac{\nu}{f}}. \qquad \text{(S-28)}$$

Numerically, we have for $f<f_0=10\nu$

$$|\epsilon_m(t,f)| < e^{-8\pi^2\frac{\nu}{f}} < e^{-8\pi^2\frac{\nu}{f_0}} \approx 3.8\times10^{-4}. \qquad \text{(S-29)}$$

2.3.3 Bounds for $\epsilon_a(t,f)$

Since $\epsilon_a(t,f)$ is in $]-\frac{\pi}{2},\frac{\pi}{2}[$, its argument can be expressed from (S-25) and (S-18) in terms of the tangent function, yielding

$$\tan\left[\epsilon_a(t,f)\right] = \frac{e^{-8\pi^2\frac{\nu}{f}}\sin(2\phi+4\pi\nu t)}{1+e^{-8\pi^2\frac{\nu}{f}}\cos(2\phi+4\pi\nu t)}.$$

The variations of the right-hand side of the expression as a function of t can be investigated (see §1.1 of Supplementary Material #3), showing that

$$|\tan\left[\epsilon_a(t,f)\right]| \le \frac{e^{-8\pi^2\frac{\nu}{f}}}{\sqrt{1-e^{-16\pi^2\frac{\nu}{f}}}},$$

or, equivalently,

$$|\epsilon_a(t,f)| < \arctan\left[\frac{e^{-8\pi^2\frac{\nu}{f}}}{\sqrt{1-e^{-16\pi^2\frac{\nu}{f}}}}\right]. \qquad \text{(S-30)}$$

Since the upper bound is an increasing function of f (see §1.2 of Supplementary Material #3), we obtain in particular that, for $f<f_0$,

$$|\epsilon_a(t,f)| < \arctan\left[\frac{e^{-8\pi^2\frac{\nu}{f_0}}}{\sqrt{1-e^{-16\pi^2\frac{\nu}{f_0}}}}\right] \approx 3.8\times10^{-4}. \qquad \text{(S-31)}$$

2.3.4 Modulus of $T_x(t,f)$

According to (S-20), (S-22), and (S-24), the modulus of $T_x(t,f)$ is given by

$$|T_x(t,f)| = \frac{\Omega}{2} e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu}{f}\right)^2} \left[1+\epsilon_m(t,f)\right], \tag{S-32}$$

with, according to (S-28),

$$|\epsilon_m(t,f)| < 1 \tag{S-33}$$

and, using (S-29) for $f < 10\nu$,

$$|\epsilon_m(t,f)| < 3.8 \times 10^{-4}. \tag{S-34}$$

Note that, for $f \geq 10\nu$, we have

$$\left|T_x^\dagger(t,f)\right| < 1.1 \times 10^{-7}\frac{\Omega}{2}, \tag{S-35}$$

which, together with (S-20) the fact that $\epsilon_m(t,f) \in]0,2[$, yields the following upper bound

$$|T_x(t,f)| < 2.2 \times 10^{-7}\,\frac{\Omega}{2}. \tag{S-36}$$

As a consequence, values of modulus are not relevant for $f \geq 10\nu$.

2.3.5 Argument of $T_x(t,f)$

According to (S-21), (S-23), and (S-25), the argument of $T_x(t,f)$ is given by

$$\arg\left[T_x(t,f)\right] = \phi - 2\pi(f-\nu)t + \epsilon_a(t,f), \tag{S-37}$$

with, according to (S-30)

$$|\epsilon_a(t,f)| < \arctan\left[\frac{e^{-8\pi^2\frac{\nu}{f}}}{\sqrt{1-e^{-16\pi^2\frac{\nu}{f}}}}\right]$$

and, using (S-31) for $f \geq 10\nu$,

$$|\epsilon_a(t,f)| < 3.8 \times 10^{-4}. \tag{S-38}$$

2.3.6 Summary of bounds

Since we are interested in $f > 0$, we always have $|\epsilon(t,f)| < 1$. Furthermore, for $f < f_0 = 10\nu$, $|\epsilon(t,f)| < 3.8 \times 10^{-4}$. For $f \geq 10\nu$, $|\epsilon(t,f)|$ can be larger (up to the upper bound of 1, reached for $f \to \infty$), but $T_x^\dagger(t,f)$ itself is then negligible, as $|T_x^\dagger(t,f)| < 1.1 \times 10^{-7}\,\Omega/2$.

For the amplitude, $|\epsilon_m(t,f)| < e^{-8\pi^2\frac{\nu}{f}}$ and, for $f < f_0$, $|\epsilon_m(t,f)| < 3.8 \times 10^{-4}$. For the phase, $|\tan\epsilon_a(t,f)| < e^{-8\pi^2\frac{\nu}{f}}/\sqrt{1-e^{-16\pi^2\frac{\nu}{f}}}$ and, for $f < f_0$, $|\epsilon_a(t,f)| < 3.8 \times 10^{-4}$.

2.4 Model with varying amplitude and phase

To derive an asymptotic form for E(POWavg†), we first need to calculate the expectation of the time-frequency transform. Using Equation (46) of the manuscript, we obtain

$$\begin{aligned}
\mathrm{E}\left[T_{x_n}^\dagger(t,f)\right] &= \frac{1}{2}e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu_0}{f}\right)^2} e^{-2i\pi(f-\nu_0)t}\mathrm{E}\left(\Omega_n e^{i\phi_n}\right) \\
&= \frac{1}{2}e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu_0}{f}\right)^2} e^{-2i\pi(f-\nu_0)t}\,\mathrm{E}\left(\Omega_n\right)\mathrm{E}\left(e^{i\phi_n}\right) \\
&= \frac{\Omega_0\rho}{2}e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu_0}{f}\right)^2} e^{i[\phi_0-2\pi(f-\nu_0)t]},
\end{aligned}$$

whose power is given by

$$\left|\mathrm{E}\left[T_x^\dagger(t,f)\right]\right|^2 = \left[\frac{\Omega_0\rho}{2}e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu_0}{f}\right)^2}\right]^2. \tag{S-39}$$

Equation (23) of the manuscript then implies

$$\mathrm{E}\left(\mathrm{POWavg}^{\dagger}\right) = \left[\frac{\Omega_0\rho}{2}e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu_0}{f}\right)^2}\right]^2 + O\left(\frac{1}{N}\right). \tag{S-40}$$

2.5 Model with varying amplitude, frequency, and phase

2.5.1 Preliminary results

We first provide the expectation of three quantities:

$$\begin{aligned} E_1(t,f) &= \mathrm{E}\left[e^{-2i\pi(f-\nu)t}\right] && \text{(S-41)}\\ E_2(\alpha,f) &= \mathrm{E}\left[e^{-\frac{1}{2}(2\pi\alpha)^2(f-\nu)^2}\right] && \text{(S-42)}\\ E_3(t,\alpha,f) &= \mathrm{E}\left[e^{-\frac{1}{2}(2\pi\alpha)^2(f-\nu)^2-2i\pi(f-\nu)t}\right], && \text{(S-43)} \end{aligned}$$

when ν is normally distributed with mean ν_0 and variance τ_ν^2.

Calculation of $E_1(t,f)$. The first quantity is given by

$$\begin{aligned} E_1(t,f) &= \frac{1}{\sqrt{2\pi\tau_\nu^2}}\int_{-\infty}^{+\infty} e^{-2i\pi(f-\nu)t}e^{-\frac{1}{2\tau_\nu^2}(\nu-\nu_0)^2}\,\mathrm{d}\nu\\ &= \frac{1}{\sqrt{2\pi\tau_\nu^2}}e^{-2i\pi ft}\int_{-\infty}^{+\infty} e^{2i\pi\nu t}e^{-\frac{1}{2\tau_\nu^2}(\nu-\nu_0)^2}\,\mathrm{d}\nu\\ &= \frac{1}{\sqrt{2\pi\tau_\nu^2}}e^{-2i\pi ft}\int_{-\infty}^{+\infty} e^{-2i\pi\nu(-t)}e^{-\frac{1}{2\tau_\nu^2}(\nu-\nu_0)^2}\,\mathrm{d}\nu\\ &= e^{-2i\pi ft}I_1(\nu_0,\tau_\nu,-t)\\ &= e^{-2i\pi ft}e^{-\frac{1}{2}(2\pi\tau_\nu t)^2}e^{2i\pi\nu_0 t}\\ &= e^{-\frac{1}{2}(2\pi\tau_\nu t)^2}e^{-2i\pi(f-\nu_0)t}, \end{aligned}$$

where I_1 is defined in (S-7).

Calculation of $E_2(\alpha,f)$. The second quantity is given by

$$E_2(\alpha,f) = \frac{1}{\sqrt{2\pi\tau_\nu^2}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(2\pi\alpha)^2(f-\nu)^2}e^{-\frac{1}{2\tau_\nu^2}(\nu-\nu_0)^2}\,\mathrm{d}\nu.$$

We reorganize the quadratic terms in the exponential

$$\begin{aligned} Q &= (2\pi\alpha)^2(f-\nu)^2 + \frac{1}{\tau_\nu^2}(\nu-\nu_0)^2\\ &= \left[(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}\right]\nu^2 - 2\nu\left[(2\pi\alpha)^2 f+\frac{\nu_0}{\tau_\nu^2}\right]\\ &\qquad +(2\pi\alpha)^2f^2+\frac{\nu_0^2}{\tau_\nu^2}. \end{aligned}$$

Setting

$$\hat{\nu} = \frac{(2\pi\alpha)^2 f+\frac{\nu_0}{\tau_\nu^2}}{(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}},$$

we obtain

$$Q = \left[(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}\right](\nu-\nu_0)^2 + (2\pi\alpha)^2f^2+\frac{\nu_0^2}{\tau_\nu^2} - \left[(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}\right]\hat{\nu}^2.$$

The term that does not depend on ν can be expanded and simplified to yield

$$(2\pi\alpha)^2 f^2 + \frac{\nu_0^2}{\tau_\nu^2} - \left[(2\pi\alpha)^2 + \frac{1}{\tau_\nu^2}\right]\hat{\nu}^2 = \frac{(2\pi\alpha)^2 \frac{1}{\tau_\nu^2}}{(2\pi\alpha)^2 + \frac{1}{\tau_\nu^2}}(f-\nu_0)^2 = \frac{(2\pi\alpha)^2}{(2\pi\alpha)^2\tau_\nu^2 + 1}(f-\nu_0)^2,$$

so that

$$Q = \left[(2\pi\alpha)^2 + \frac{1}{\tau_\nu^2}\right](\nu-\nu_0)^2 + \frac{(2\pi\alpha)^2}{(2\pi\alpha)^2\tau_\nu^2 + 1}(f-\nu_0)^2$$

and

$$E_2(\alpha, f) = \frac{1}{\sqrt{2\pi\tau_\nu^2}} e^{-\frac{1}{2}\frac{(2\pi\alpha)^2}{(2\pi\alpha)^2\tau_\nu^2+1}(f-\nu_0)^2} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}\right](\nu-\nu_0)^2} \mathrm{d}\nu. \tag{S-44}$$

The integral can be calculated using the fact that a normal distribution sums to 1 (Polyanin and Manzhirov, 2007, Equation (20.2.4.5)), leading to

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}\right](\nu-\nu_0)^2} \mathrm{d}\nu = \sqrt{\frac{2\pi}{(2\pi\alpha)^2 + \frac{1}{\tau_\nu^2}}}$$

and

$$E_2(\alpha, f) = \frac{1}{\sqrt{(2\pi\alpha)^2\tau_\nu^2 + 1}} e^{-\frac{1}{2}\frac{(2\pi\alpha)^2}{(2\pi\alpha)^2\tau_\nu^2+1}(f-\nu_0)^2}.$$

Calculation of $E_3(t, \alpha, f)$. $E_3(t, \alpha, f)$ is given by

$$E_3(t, \alpha, f) = \frac{1}{\sqrt{2\pi\tau_\nu^2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(2\pi\alpha)^2(f-\nu)^2} e^{-\frac{1}{2\tau_\nu^2}(\nu-\nu_0)^2} e^{-2i\pi(f-\nu)t} \mathrm{d}\nu.$$

We use (S-44) to express the real term in the exponential, yielding

$$E_3(t, \alpha, f) = \frac{1}{\sqrt{2\pi\tau_\nu^2}} e^{-\frac{1}{2}\frac{(2\pi\alpha)^2}{(2\pi\alpha)^2\tau_\nu^2+1}(f-\nu_0)^2} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}\right](\nu-\nu_0)^2} e^{-2i\pi(f-\nu)t} \mathrm{d}\nu.$$

The integral in this expression rereads

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}\right](\nu-\nu_0)^2} e^{-2i\pi(f-\nu)t} \mathrm{d}\nu = e^{-2i\pi ft} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}\right](\nu-\nu_0)^2} e^{2i\pi\nu t} \mathrm{d}\nu.$$

Performing the parameter change $\xi = -\nu$, we obtain

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}\right](\nu-\nu_0)^2} e^{2i\pi\nu t} \mathrm{d}\nu = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}\right](\xi+\nu_0)^2} e^{-2i\pi\xi t} \mathrm{d}\xi.$$

This integral can be calculated using (S-8), leading to

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}\right](\xi+\nu_0)^2} e^{-2i\pi\xi t} \mathrm{d}\xi = \sqrt{\frac{2\pi\tau_\nu^2}{(2\pi\alpha)^2\tau_\nu^2 + 1}} e^{-\frac{1}{2}\frac{(2\pi t)^2}{(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}}} e^{2i\pi\nu_0 t}.$$

We therefore have

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left[(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}\right](\nu-\nu_0)^2} e^{-2i\pi(f-\nu)t} \mathrm{d}\nu = \sqrt{\frac{2\pi\tau_\nu^2}{(2\pi\alpha)^2\tau_\nu^2 + 1}} e^{-\frac{1}{2}\frac{(2\pi t)^2}{(2\pi\alpha)^2+\frac{1}{\tau_\nu^2}}} e^{-2i\pi(f-\nu_0)t}$$

and

$$E_3(t, \alpha, f) = \frac{1}{\sqrt{(2\pi\alpha)^2\tau_\nu^2 + 1}} e^{-\frac{1}{2}\frac{(2\pi\alpha)^2}{(2\pi\alpha)^2\tau_\nu^2+1}(f-\nu_0)^2} e^{-\frac{1}{2}\frac{(2\pi\tau_\nu)^2}{(2\pi\alpha)^2\tau_\nu^2+1}t^2} e^{-2i\pi(f-\nu_0)t} = \frac{1}{\sqrt{(2\pi\alpha)^2\tau_\nu^2 + 1}} e^{-2i\pi(f-\nu_0)t} e^{-\frac{1}{2}\frac{(2\pi\alpha)^2}{(2\pi\alpha)^2\tau_\nu^2+1}\left[(f-\nu_0)^2+\left(\frac{\tau_\nu}{\alpha}\right)^2 t^2\right]}.$$

2.5.2 Expectation of $|T_x^\dagger(t,f)|$

With the definition of $T_x^\dagger(t,f)$ from Equation (28) from the manuscript, we obtain that

$$\begin{aligned} \mathrm{E}\left[\left|T_x^\dagger(t,f)\right|\right] &= \mathrm{E}\left[\frac{\Omega}{2} e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu}{f}\right)^2}\right] \\ &= \mathrm{E}\left[\frac{\Omega}{2}\right] \mathrm{E}\left[e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu}{f}\right)^2}\right], \end{aligned}$$

since Ω and ν are independent. The first expectation in the right-hand is straightforward to calculate, yielding

$$\mathrm{E}\left[\frac{\Omega}{2}\right] = \frac{\Omega_0}{2}. \tag{S-45}$$

The second expectation in the right-hand side needs to be calculated explicitly as

$$\begin{aligned} \mathrm{E}\left[e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu}{f}\right)^2}\right] &= \int e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu}{f}\right)^2} \mathrm{p}(\nu)\, \mathrm{d}\nu \\ &= \frac{1}{\sqrt{2\pi\tau_\nu^2}} \int e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu}{f}\right)^2} e^{-\frac{1}{2\tau_\nu^2}(\nu-\nu_0)^2}\, \mathrm{d}\nu \\ &= E_2\left(\frac{1}{f}, f\right) \\ &= \frac{1}{\sqrt{\left(\frac{2\pi\tau_\nu^2}{f}\right)^2+1}} e^{-\frac{1}{2}\frac{(2\pi)^2}{\left(\frac{2\pi\tau_\nu^2}{f}\right)^2+1}\left(1-\frac{\nu_0}{f}\right)^2}. \end{aligned} \tag{S-46}$$

2.5.3 Expectation of $e^{i\arg\left[T_x^\dagger(t,f)\right]}$

We have

$$\begin{aligned} \mathrm{E}\left\{e^{i\arg\left[T_x^\dagger(t,f)\right]}\right\} &= \mathrm{E}\left\{e^{i[\phi-2\pi(f-\nu)t]}\right\} \\ &= \mathrm{E}\left(e^{i\phi}\right) \mathrm{E}\left[e^{-2i\pi(f-\nu)t}\right], \end{aligned} \tag{S-47}$$

since ϕ and ν are independent. According to Equation (35) of the manuscript, the value of the first expectation of the right-hand side is given by $\rho e^{i\phi_0}$. As to the second expectation, it yields

$$\begin{aligned} \mathrm{E}\left[e^{-2i\pi(f-\nu)t}\right] &= E_1(t,f) \\ &= e^{-\frac{1}{2}(2\pi\tau_\nu t)^2} e^{-2i\pi(f-\nu_0)t}. \end{aligned} \tag{S-48}$$

2.5.4 Expectation of $T_x^\dagger(t,f)$

We have

$$\begin{aligned} \mathrm{E}\left[T_x^\dagger(t,f)\right] &= \mathrm{E}\left\{\frac{\Omega}{2} e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu}{f}\right)^2} e^{i[\phi-2\pi(f-\nu)t]}\right\} \\ &= \mathrm{E}\left(\frac{\Omega}{2}\right) \mathrm{E}\left(e^{i\phi}\right) \mathrm{E}\left[e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu}{f}\right)^2-2i\pi(f-\nu)t}\right], \end{aligned} \tag{S-49}$$

with $\mathrm{E}(\Omega/2) = \Omega_0/2$, $\mathrm{E}(e^{i\phi}) = \rho e^{i\phi_0}$, and

$$\begin{aligned}\mathrm{E}\left[e^{-\frac{1}{2}(2\pi)^2\left(1-\frac{\nu}{f}\right)^2-2i\pi(f-\nu)t}\right] &= E_3\left(t,\frac{1}{f},f\right) \\ &= \frac{1}{\sqrt{\left(\frac{2\pi\tau_\nu}{f}\right)^2+1}} e^{-2i\pi(f-\nu_0)t}\, e^{-\frac{1}{2}\frac{(2\pi)^2}{\left(\frac{2\pi\tau_\nu}{f}\right)^2+1}\left[\left(1-\frac{\nu_0}{f}\right)^2+\tau_\nu^2 t^2\right]}. \end{aligned} \tag{S-50}$$

2.6 Summary of results

The results regarding the S-transform of an oscillatory model of increasing complexity are summarized here.

Model	Section	Distributions	Relationship between quantities
varying ϕ	§II-D4	$\phi_n \sim \mathrm{VonMises}(\phi_0,\kappa)$; $\Omega_n = \Omega_0$; $\nu_n = \nu_0$	$\mathrm{POWavg} = \mathrm{avgAMP}^2 \times \mathrm{ITC}^2$
varying ϕ and Ω	§II-D5	$\phi_n \sim \mathrm{VonMises}(\phi_0,\kappa)$; $\Omega_n \sim \mathcal{N}(\Omega_0,\tau_\Omega^2)$; $\nu_n = \nu_0$	$\mathrm{E}[\mathrm{POWavg} - \mathrm{avgAMP}^2 \times \mathrm{ITC}^2] = O\left(\frac{1}{N}\right)$
varying ϕ, Ω, and ν	§II-D6	$\phi_n \sim \mathrm{VonMises}(\phi_0,\kappa)$; $\Omega_n \sim \mathcal{N}(\Omega_0,\tau_\Omega^2)$; $\nu_n \sim \mathcal{N}(\nu_0,\tau_\nu^2)$	Nontrivial, see Equation (70)

3 Proof of general relationship between avgAMP, ITC, and POWavg

We here provide a sketch of proof. Detailed results can be found in §2 of Supplementary Material #3. We expand $\mathrm{avgAMP}^2 \times \mathrm{ITC}^2$ from (S-1) and (S-3), yielding

$$\begin{aligned}\mathrm{avgAMP}_{x_{1:N}}(t,f)^2 \times \mathrm{ITC}_{x_{1:N}}(t,f)^2 &= \underbrace{\frac{1}{N^3}\sum_{k=1}^{N}|T_{x_k}(t,f)|^2}_{S_1} + \underbrace{\frac{1}{N^3}\sum_{k\neq l}|T_{x_k}(t,f)|\,|T_{x_l}(t,f)|}_{S_2} \\ &\quad \underbrace{+\frac{1}{N^4}\left\{\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\right\}\left[\sum_{k=1}^{N}|T_{x_k}(t,f)|^2\right]}_{P_1} \\ &\quad \underbrace{+\frac{1}{N^4}\left\{\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\right\} \times \left[\sum_{k\neq l}|T_{x_k}(t,f)|\,|T_{x_l}(t,f)|\right]}_{P_2}. \end{aligned} \tag{S-51}$$

P_1 of (S-51) can be further expanded, yielding

$$\begin{aligned} P_1 &= \underbrace{\frac{1}{N^4}\sum_{m\neq n}|T_{x_m}(t,f)|^2\, e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}}_{S_3} \\ &\quad \underbrace{+\frac{1}{N^4}\sum_{m\neq n}|T_{x_n}(t,f)|^2\, e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}}_{S_4} \\ &\quad \underbrace{+\frac{1}{N^4}\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{k\notin\{m,n\}}|T_{x_k}(t,f)|^2}_{S_5}. \end{aligned} \tag{S-52}$$

P_2 of (S-51) can also be expanded:

$$\begin{aligned} P_2 &= \underbrace{\frac{2}{N^4}\sum_{m\neq n}|T_{x_m}(t,f)|\,|T_{x_n}(t,f)|\, e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}}_{S_6} \\ &\quad \underbrace{+\frac{2}{N^4}\sum_{m\neq n}|T_{x_m}(t,f)|\, e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|}_{S_7} \\ &\quad \underbrace{+\frac{2}{N^4}\sum_{m\neq n}|T_{x_n}(t,f)|\, e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|}_{S_8} \\ &\quad \underbrace{+\frac{1}{N^4}\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]} \times \sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|\sum_{k\notin\{l,m,n\}}|T_{x_k}(t,f)|}_{S_9}. \end{aligned} \tag{S-53}$$

We were able to expand $\mathrm{avgAMP}_{x_{1:N}}(t,f)^2 \times \mathrm{ITC}_{x_{1:N}}(t,f)^2$ into 9 terms: two (S_1 and S_2) from (S-51), three (S_3 to S_5) from (S-52), and 4 (S_6 to S_9) from (S-53). We can now calculate the expectation of $\mathrm{avgAMP}_{x_{1:N}}(t,f)^2 \times \mathrm{ITC}_{x_{1:N}}(t,f)^2$ term by term.

$$\mathrm{E}(S_1) = \frac{1}{N^2}\mathrm{E}\left[|T_x(t,f)|^2\right],$$

for a global contribution that is $O(1/N^2)$;

$$\mathrm{E}(S_2) = \frac{N-1}{N^2}\mathrm{E}\left[|T_x(t,f)|\right]^2,$$

for a global contribution that is $O(1/N)$;

$$\mathrm{E}(S_3) = \frac{N-1}{N^3}\mathrm{E}\left[|T_x(t,f)|^2\, e^{i\theta_x(t,f)}\right]\mathrm{E}\left[e^{i\theta_x(t,f)}\right]^*,$$

which is $O(1/N^2)$;

$$\mathrm{E}(S_4) = \frac{N-1}{N^3}\mathrm{E}\left[|T_x(t,f)|^2\, e^{-i\theta_x(t,f)}\right]\mathrm{E}\left[e^{i\theta_x(t,f)}\right],$$

which is also $O(1/N^2)$;

$$\mathrm{E}(S_5) = \frac{(N-1)(N-2)}{N^3}\mathrm{E}\left[|T_x(t,f)|^2\right]\left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right|^2,$$

which is $O(1/N)$;

$$\mathrm{E}(S_6) = \frac{2(N-1)}{N^3}\left|\mathrm{E}\left[T_x(t,f)\right]\right|^2,$$

which is $O(1/N^2)$;

$$\mathrm{E}(S_7) = \frac{2(N-1)(N-2)}{N^3}\mathrm{E}\left[T_x(t,f)\right]\mathrm{E}\left[e^{i\theta_x(t,f)}\right]^*\mathrm{E}\left[|T_x(t,f)|\right],$$

which is $O(1/N)$;

$$\mathrm{E}(S_8) = \frac{2(N-1)(N-2)}{N^3}\mathrm{E}\left[T_x(t,f)\right]^*\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\mathrm{E}\left[|T_x(t,f)|\right],$$

which is $O(1/N)$;

$$\mathrm{E}(S_9) = \frac{(N-1)(N-2)(N-3)}{N^3}\left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right|^2\mathrm{E}\left[|T_x(t,f)|\right]^2,$$

which is the only term to be $O(1)$. Putting all expressions together, we are led to

$$\mathrm{E}(\mathrm{avgAMP}^2 \times \mathrm{ITC}^2) = \left|\mathrm{E}\left[e^{i\theta_x(t,f)]}\right]\right|^2\mathrm{E}\left[|T_x(t,f)|\right]^2 + O\left(\frac{1}{N}\right). \quad \text{(S-54)}$$

We now need to express POWavg. From (S-6), we have

$$\mathrm{E}\left[\mathrm{POWavg}_{x_{1:N}}(t,f)\right] = \left|\mathrm{E}\left[T_x(t,f)\right]\right|^2 + O\left(\frac{1}{N}\right).$$

The expectation can be expressed by using Equation (11) of the manuscript,

$$\mathrm{E}\left[T_x(t,f)\right] = \mathrm{E}\left[|T_x(t,f)|\,e^{i\theta_x(t,f)}\right],$$

and further developed using Equation (6) of the manuscript,

$$\mathrm{E}\left[|T_x(t,f)|\,e^{i\theta_x(t,f)}\right] = \mathrm{E}\left[e^{i\theta_x(t,f)}\right]\mathrm{E}\left[|T_x(t,f)|\right] + \mathrm{Cov}\left[e^{i\theta_x(t,f)}, |T_x(t,f)|\right].$$

Consequently,

$$\left|\mathrm{E}\left[T_x(t,f)\right]\right|^2 = \left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\mathrm{E}\left[|T_x(t,f)|\right] + \mathrm{Cov}\left[e^{i\theta_x(t,f)}, |T_x(t,f)|\right]\right|^2. \quad \text{(S-55)}$$

As a conclusion, we have from (S-54) and (S-55) that

$$\begin{aligned} &\mathrm{E}\left[\mathrm{POWavg}_{x_{1:N}}(t,f)\right] - \mathrm{E}\left[\mathrm{ITC}_{x_{1:N}}(t,f)^2 \times \mathrm{avgAMP}_{x_{1:N}}(t,f)^2\right] \\ = &\left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\mathrm{E}\left[|T_x(t,f)|\right] + \mathrm{Cov}\left[e^{i\theta_x(t,f)}, |T_x(t,f)|\right]\right|^2 - \left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right|^2\mathrm{E}\left[|T_x(t,f)|\right]^2 + O\left(\frac{1}{N}\right). \end{aligned} \quad \text{(S-56)}$$

This is in general not $O(1/N)$. A particular case occurs when

$$\mathrm{Cov}\left[e^{i\theta_x(t,f)}, |T_x(t,f)|\right] = 0, \quad \text{(S-57)}$$

which does make the difference of (S-56) $O(1/N)$. Since independence implies a zero covariance, independence of $e^{i\theta_x(t,f)}$ and $|T_x(t,f)|$ has the same effect on (S-56). Considering more general solutions is more challenging. For instance, considering the first moments of $e^{i\theta_x(t,f)}$ and $|T_x(t,f)|$ fixed, the difference is $O(1/N)$ only if we have a relation of the form

$$|z - z_0|^2 = r^2,$$

with

$$\begin{aligned} z &= \operatorname{Cov}\left[e^{i\theta_x(t,f)}, |T_x(t,f)|\right] \\ z_0 &= -\mathrm{E}\left[e^{i\theta_x(t,f)}\right] \mathrm{E}[|T_x(t,f)|] \\ r &= \left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right| \mathrm{E}[|T_x(t,f)|]. \end{aligned}$$

The complex numbers z that respect (3) are on a circle of center z_0 and radius r. The case of zero covariance mentioned above corresponds to the case $z = 0$.

References

Polyanin, A.D., Manzhirov, A.V., 2007. Handbook of Mathematics for Engineers and Scientists. Chapman & Hall/CRC, Boca Raton.