Supplementary material #3 for manuscript "Time-frequency analysis of event-related brain recordings: Connecting power of evoked potential and inter-trial coherence"

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Contents

1 Oscillatory model

1.1 First result

We here provide bounds for the function

$$t \mapsto \frac{e^{-8\pi^2 \frac{\nu}{f}} \sin(2\phi + 4\pi\nu t)}{1 + e^{-8\pi^2 \frac{\nu}{f}} \cos(2\phi + 4\pi\nu t)}. \tag{S-1}$$

We first set $b_f = e^{-8\pi^2 \frac{\nu}{f}}$, $u = 2\phi + 4\pi\nu t$ and

$$g(u) = \frac{b_f \sin u}{1 + b_f \cos u}. \tag{S-2}$$

The derivative of g is given by

$$\begin{aligned} g'(u) &= \frac{b_f \cos u(1 + b_f \cos u) + b_f^2 \sin^2 u}{(1 + b_f \cos u)^2} \\ &= \frac{b_f^2 + b_f \cos u}{(1 + b_f \cos u)^2}. \end{aligned}$$

We therefore have $g'(u) = 0$ if and only if $\cos u = -b_f$, i.e., $u = \pm \arccos(-b_f)$. For $b_f < 1$, g' is strictly negative on $[-\pi, -\arccos(-b_f)[$ and $]\arccos(-b_f), \pi]$, and strictly positive on $]-\arccos(-b_f), \arccos(-b_f)[$. As a consequence, g is strictly decreasing on $[-\pi, -\arccos(-b_f)[$ and $]\arccos(-b_f), \pi]$, and strictly increasing on $]-\arccos(-b_f), \arccos(-b_f)[$. We also have $g(-\pi) = g(\pi) = 0$, as well as

$$\begin{aligned} g[\arccos(-b_f)] &= \frac{b_f \sin[\arccos(-b_f)]}{1 + b_f \cos[\arccos(-b_f)]} \\ &= \frac{b_f \sqrt{1 - b_f^2}}{1 - b_f^2} \\ &= \frac{b_f}{\sqrt{1 - b_f^2}} \\ g[-\arccos(-b_f)] &= \frac{b_f \sin[-\arccos(-b_f)]}{1 + b_f \cos[-\arccos(-b_f)]} \\ &= -\frac{b_f}{\sqrt{1 - b_f^2}}. \end{aligned}$$

Setting $g_f = b_f / \sqrt{1 - b_f^2}$, we obtain that $-g_f$ is a lower bound of g, and g_f an upper bound. Finally, $|g(u)| \leq g_f$ for all u.

1.2 Second result

We prove that

$$f \mapsto \arctan \frac{e^{-8\pi^2 \frac{\nu}{f}} \sqrt{1 - e^{-16\pi^2 \frac{\nu}{f}}}}{1 - e^{-16\pi^2 \frac{\nu}{f}}} \tag{S-3}$$

is a strictly increasing function of f. To show this result, we show that the function is the composition of three functions that are strictly positive. First, it is direct to see that $f \mapsto e^{-8\pi^2 \frac{\nu}{f}}$ is a strictly increasing function. We then consider

$$h(u) = \frac{u\sqrt{1-u^2}}{1-u^2}. \tag{S-4}$$

Its derivative is given by

$$\begin{aligned} h'(u) &= \frac{\sqrt{1-u^2}}{1-u^2} + \frac{u}{1-u^2}\frac{-2u}{2\sqrt{1-u^2}} + u\sqrt{1-u^2}\frac{2u}{(1-u^2)^2} \\ &= \frac{1}{\sqrt{1-u^2}} + \frac{u^2}{(1-u^2)^{\frac{3}{2}}} \\ &= \frac{1}{(1-u^2)^{\frac{3}{2}}}, \end{aligned}$$

which is positive. So h is also strictly increasing. Finally, $u \mapsto \arctan u$ is strictly increasing.

2 Proof of general relationship

We expand avgAMP^2 $\times$ ITC^2 from Equations (80) and (82) of the manuscript:

$$\begin{aligned}
&\text{avgAMP}_{x_{1:N}}(t,f)^2 \times \text{ITC}_{x_{1:N}}(t,f)^2 \\
=\;& \frac{1}{N^2}\left[\sum_{k=1}^{N}|T_{x_k}(t,f)|^2 + \sum_{k\neq l}|T_{x_k}(t,f)|\,|T_{x_l}(t,f)|\right]\left\{\frac{1}{N} + \frac{1}{N^2}\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\right\} \\
=\;& \underbrace{\frac{1}{N^3}\sum_{k=1}^{N}|T_{x_k}(t,f)|^2}_{S_1} + \underbrace{\frac{1}{N^3}\sum_{k\neq l}|T_{x_k}(t,f)|\,|T_{x_l}(t,f)|}_{S_2} + \underbrace{\frac{1}{N^4}\left[\sum_{k=1}^{N}|T_{x_k}(t,f)|^2\right]\left\{\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\right\}}_{P_1} \\
&+ \underbrace{\frac{1}{N^4}\left[\sum_{k\neq l}|T_{x_k}(t,f)|\,|T_{x_l}(t,f)|\right]\left\{\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\right\}}_{P_2}.
\end{aligned} \tag{S-5}$$

P_1 of Equation (S-5) needs to be further expanded, yielding

$$\begin{aligned}
P_1 &= \frac{1}{N^4}\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{k=1}^{N}|T_{x_k}(t,f)|^2 \\
&= \frac{1}{N^4}\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\left[|T_{x_m}(t,f)|^2 + |T_{x_n}(t,f)|^2 + \sum_{k\notin\{m,n\}}|T_{x_k}(t,f)|^2\right] \\
&= \underbrace{\frac{1}{N^4}\sum_{m\neq n}|T_{x_m}(t,f)|^2\, e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}}_{S_3} + \underbrace{\frac{1}{N^4}\sum_{m\neq n}|T_{x_n}(t,f)|^2\, e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}}_{S_4} + \underbrace{\frac{1}{N^4}\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{k\notin\{m,n\}}|T_{x_k}(t,f)|^2}_{S_5}.
\end{aligned} \tag{S-6}$$

P_2 of Equation (S-5) also needs to be expanded:

$$\begin{aligned}
P_2 &= \frac{1}{N^4}\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{k\neq l}|T_{x_k}(t,f)|\,|T_{x_l}(t,f)| \\
&= \frac{1}{N^4}\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{k=1}^{N}|T_{x_k}(t,f)|\sum_{l\neq k}|T_{x_l}(t,f)| \\
&= \frac{1}{N^4}\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\left[|T_{x_m}(t,f)|\sum_{l\neq m}|T_{x_l}(t,f)|+|T_{x_n}(t,f)|\sum_{l\neq n}|T_{x_l}(t,f)|+\sum_{k\notin\{m,n\}}|T_{x_k}(t,f)|\sum_{l\neq k}|T_{x_l}(t,f)|\right] \\
&= \frac{1}{N^4}\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\left\{|T_{x_m}(t,f)|\,|T_{x_n}(t,f)|+|T_{x_m}(t,f)|\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|+|T_{x_n}(t,f)|\,|T_{x_m}(t,f)|+|T_{x_n}(t,f)|\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|\right. \\
&\qquad\left.+\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|\left[|T_{x_m}(t,f)|+|T_{x_n}(t,f)|+\sum_{k\notin\{l,m,n\}}|T_{x_k}(t,f)|\right]\right\} \\
&= \frac{1}{N^4}\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\left[2\,|T_{x_m}(t,f)|\,|T_{x_n}(t,f)|+2\left(|T_{x_m}(t,f)|+|T_{x_n}(t,f)|\right)\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|+\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|\sum_{k\notin\{l,m,n\}}|T_{x_k}(t,f)|\right] \\
&= \underbrace{\frac{2}{N^4}\sum_{m\neq n}|T_{x_m}(t,f)|\,|T_{x_n}(t,f)|\,e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}}_{S_6}+\underbrace{\frac{2}{N^4}\sum_{m\neq n}|T_{x_m}(t,f)|\,e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|}_{S_7} \\
&\qquad+\underbrace{\frac{2}{N^4}\sum_{m\neq n}|T_{x_n}(t,f)|\,e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|}_{S_8}+\underbrace{\frac{1}{N^4}\sum_{m\neq n}e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|\sum_{k\notin\{l,m,n\}}|T_{x_k}(t,f)|}_{S_9}\,. \qquad \text{(S-7)}
\end{aligned}$$

As a summary, we were able to expand $\text{avgAMP}_{x_{1:N}}(t,f)^2 \times \text{ITC}_{x_{1:N}}(t,f)^2$ into 9 terms: two (S_1 and S_2) from Equation (S-5), three (S_3 to S_5) from Equation (S-6), and 4 (S_6 to S_9) from Equation (S-7). We can now calculate the expectation of $\text{avgAMP}_{x_{1:N}}(t,f)^2 \times \text{ITC}_{x_{1:N}}(t,f)^2$ term by term:

- The first term, $\mathrm{E}(S_1)$, yields

$$\begin{aligned}\mathrm{E}(S_1) &= \mathrm{E}\left[\frac{1}{N^3}\sum_{k=1}^{N}|T_{x_k}(t,f)|^2\right]\\ &= \frac{1}{N^3}\sum_{k=1}^{N}\mathrm{E}\left[|T_{x_k}(t,f)|^2\right]\\ &= \frac{1}{N^3}\sum_{k=1}^{N}\mathrm{E}\left[|T_{x}(t,f)|^2\right]\\ &= \frac{1}{N^2}\mathrm{E}\left[|T_{x}(t,f)|^2\right],\end{aligned}$$

  for a resulting contribution to the general sum that is $O(1/N^2)$.

- The second term, $\mathrm{E}(S_2)$, is equal to

$$\begin{aligned}\mathrm{E}(S_2) &= \mathrm{E}\left[\frac{1}{N^3}\sum_{k\neq l}|T_{x_k}(t,f)|\,|T_{x_l}(t,f)|\right]\\ &= \frac{1}{N^3}\sum_{k\neq l}\mathrm{E}\left[|T_{x_k}(t,f)|\,|T_{x_l}(t,f)|\right]\\ &= \frac{1}{N^3}\sum_{k\neq l}\mathrm{E}\left[|T_{x_k}(t,f)|\right]\mathrm{E}\left[|T_{x_l}(t,f)|\right]\\ &= \frac{1}{N^3}\sum_{k\neq l}\mathrm{E}\left[|T_{x}(t,f)|\right]^2\\ &= \frac{N-1}{N^2}\mathrm{E}\left[|T_{x}(t,f)|\right]^2,\end{aligned}$$

  for a global contribution that is $O(1/N)$.

- The third term, $\mathrm{E}(S_3)$, is given by

$$\begin{aligned}
\mathrm{E}(S_3) &= \mathrm{E}\left\{\frac{1}{N^4}\sum_{m\neq n}|T_{x_m}(t,f)|^2\, e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\right\} \\
&= \frac{1}{N^4}\sum_{m\neq n}\mathrm{E}\left\{|T_{x_m}(t,f)|^2\, e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\right\} \\
&= \frac{1}{N^4}\sum_{m\neq n}\mathrm{E}\left[|T_{x_m}(t,f)|^2\, e^{i\theta_{x_m}(t,f)}\right]\mathrm{E}\left[e^{-i\theta_{x_n}(t,f)}\right] \\
&= \frac{1}{N^4}\sum_{m\neq n}\mathrm{E}\left[|T_{x}(t,f)|^2\, e^{i\theta_{x}(t,f)}\right]\mathrm{E}\left[e^{-i\theta_{x}(t,f)}\right] \\
&= \frac{N-1}{N^3}\mathrm{E}\left[|T_{x}(t,f)|^2\, e^{i\theta_{x}(t,f)}\right]\mathrm{E}\left[e^{i\theta_{x}(t,f)}\right]^*,
\end{aligned}$$

  which is $O(1/N^2)$.

- The fourth term, $\mathrm{E}(S_4)$, is similar to the previous,

$$\begin{aligned}
\mathrm{E}(S_4) &= \mathrm{E}\left\{\frac{1}{N^4}\sum_{m\neq n}|T_{x_n}(t,f)|^2\, e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\right\} \\
&= \frac{N-1}{N^3}\mathrm{E}\left[|T_{x}(t,f)|^2\, e^{i\theta_{x}(t,f)}\right]^*\mathrm{E}\left[e^{i\theta_{x}(t,f)}\right],
\end{aligned}$$

  which is also $O(1/N^2)$.

- The fifth term, $\mathrm{E}(S_5)$, yields

$$\begin{aligned}
\mathrm{E}(S_5) &= \mathrm{E}\left\{ \frac{1}{N^4} \sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]} \sum_{k\notin\{m,n\}} |T_{x_k}(t,f)|^2 \right\} \\
&= \frac{1}{N^4} \sum_{m\neq n} \mathrm{E}\left\{ e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]} \sum_{k\notin\{m,n\}} |T_{x_k}(t,f)|^2 \right\} \\
&= \frac{1}{N^4} \sum_{m\neq n} \mathrm{E}\left[e^{i\theta_{x_m}(t,f)}\right] \mathrm{E}\left[e^{-i\theta_{x_n}(t,f)}\right] \mathrm{E}\left[\sum_{k\notin\{m,n\}} |T_{x_k}(t,f)|^2 \right] \\
&= \frac{1}{N^4} \sum_{m\neq n} \mathrm{E}\left[e^{i\theta_{x_m}(t,f)}\right] \mathrm{E}\left[e^{i\theta_{x_n}(t,f)}\right]^* \sum_{k\notin\{m,n\}} \mathrm{E}\left[|T_{x_k}(t,f)|^2\right] \\
&= \frac{1}{N^4} \sum_{m\neq n} \mathrm{E}\left[e^{i\theta_{x}(t,f)}\right] \mathrm{E}\left[e^{i\theta_{x}(t,f)}\right]^* \sum_{k\notin\{m,n\}} \mathrm{E}\left[|T_{x}(t,f)|^2\right] \\
&= \frac{N(N-1)(N-2)}{N^4} \mathrm{E}\left[|T_x(t,f)|^2\right] \left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right|^2 \\
&= \frac{(N-1)(N-2)}{N^3} \mathrm{E}\left[|T_x(t,f)|^2\right] \left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right|^2,
\end{aligned}$$

which is $O(1/N)$.

- The sixth term, $\mathrm{E}(S_6)$, yields

$$
\begin{aligned}
S_6 &= \mathrm{E}\left\{\frac{2}{N^4}\sum_{m\neq n}|T_{x_m}(t,f)|\,|T_{x_n}(t,f)|\,e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\right\} \\
&= \frac{2}{N^4}\sum_{m\neq n}\mathrm{E}\left\{|T_{x_m}(t,f)|\,|T_{x_n}(t,f)|\,e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\right\} \\
&= \frac{2}{N^4}\sum_{m\neq n}\mathrm{E}\left[|T_{x_m}(t,f)|\,e^{i\theta_{x_m}(t,f)}\right]\mathrm{E}\left[|T_{x_n}(t,f)|\,e^{-i\theta_{x_n}(t,f)}\right] \\
&= \frac{2}{N^4}\sum_{m\neq n}\mathrm{E}\left[T_{x_m}(t,f)\right]\mathrm{E}\left[T_{x_n}(t,f)^*\right] \\
&= \frac{2}{N^4}\sum_{m\neq n}\mathrm{E}\left[T_x(t,f)\right]\mathrm{E}\left[T_x(t,f)\right]^* \\
&= \frac{2}{N^4}\sum_{m\neq n}\left|\mathrm{E}\left[T_x(t,f)\right]\right|^2 \\
&= \frac{2N(N-1)}{N^4}\left|\mathrm{E}\left[T_x(t,f)\right]\right|^2 \\
&= \frac{2(N-1)}{N^3}\left|\mathrm{E}\left[T_x(t,f)\right]\right|^2,
\end{aligned}
$$

which is $O(1/N^2)$.

- The seventh term, $\mathrm{E}(S_7)$, yields

$$\begin{aligned}
\mathrm{E}(S_7) &= \mathrm{E}\left\{\frac{2}{N^4}\sum_{m\neq n}|T_{x_m}(t,f)|\,e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|\right\}\\
&= \frac{2}{N^4}\sum_{m\neq n}\mathrm{E}\left\{|T_{x_m}(t,f)|\,e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|\right\}\\
&= \frac{2}{N^4}\sum_{m\neq n}\mathrm{E}\left[|T_{x_m}(t,f)|\,e^{i\theta_{x_m}(t,f)}\right]\mathrm{E}\left[e^{-i\theta_{x_n}(t,f)}\right]\mathrm{E}\left[\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|\right]\\
&= \frac{2}{N^4}\sum_{m\neq n}\mathrm{E}\left[T_{x_m}(t,f)\right]\mathrm{E}\left[e^{i\theta_{x_n}(t,f)}\right]^*\sum_{l\notin\{m,n\}}\mathrm{E}\left[|T_{x_l}(t,f)|\right]\\
&= \frac{2}{N^4}\sum_{m\neq n}\mathrm{E}\left[T_x(t,f)\right]\mathrm{E}\left[e^{i\theta_x(t,f)}\right]^*\sum_{l\notin\{m,n\}}\mathrm{E}\left[|T_x(t,f)|\right]\\
&= \frac{2N(N-1)(N-2)}{N^4}\mathrm{E}\left[T_x(t,f)\right]\mathrm{E}\left[e^{i\theta_x(t,f)}\right]^*\mathrm{E}\left[|T_x(t,f)|\right]\\
&= \frac{2(N-1)(N-2)}{N^3}\mathrm{E}\left[T_x(t,f)\right]\mathrm{E}\left[e^{i\theta_x(t,f)}\right]^*\mathrm{E}\left[|T_x(t,f)|\right],
\end{aligned}$$

which is $O(1/N)$.

- The eight term, $\mathrm{E}(S_8)$, yields

$$
\begin{aligned}
\mathrm{E}(S_8) &= \mathrm{E}\left\{ \frac{2}{N^4} \sum_{m \neq n} |T_{x_n}(t,f)| \, e^{i[\theta_{x_m}(t,f) - \theta_{x_n}(t,f)]} \sum_{l \notin \{m,n\}} |T_{x_l}(t,f)| \right\} \\
&= \frac{2}{N^4} \sum_{m \neq n} \mathrm{E}\left\{ |T_{x_n}(t,f)| \, e^{i[\theta_{x_m}(t,f) - \theta_{x_n}(t,f)]} \sum_{l \notin \{m,n\}} |T_{x_l}(t,f)| \right\} \\
&= \frac{2}{N^4} \sum_{m \neq n} \mathrm{E}\left[|T_{x_n}(t,f)| \, e^{-i\theta_{x_n}(t,f)} \right] \mathrm{E}\left[e^{i\theta_{x_m}(t,f)} \right] \mathrm{E}\left[\sum_{l \notin \{m,n\}} |T_{x_l}(t,f)| \right] \\
&= \frac{2}{N^4} \sum_{m \neq n} \mathrm{E}\left[T_{x_n}(t,f)^* \right] \mathrm{E}\left[e^{i\theta_{x_m}(t,f)} \right] \sum_{l \notin \{m,n\}} \mathrm{E}\left[|T_{x_l}(t,f)| \right] \\
&= \frac{2}{N^4} \sum_{m \neq n} \mathrm{E}\left[T_x(t,f) \right]^* \mathrm{E}\left[e^{i\theta_x(t,f)} \right] \sum_{l \notin \{m,n\}} \mathrm{E}\left[|T_x(t,f)| \right] \\
&= \frac{2N(N-1)(N-2)}{N^4} \mathrm{E}\left[T_x(t,f) \right]^* \mathrm{E}\left[e^{i\theta_x(t,f)} \right] \mathrm{E}\left[|T_x(t,f)| \right] \\
&= \frac{2(N-1)(N-2)}{N^3} \mathrm{E}\left[T_x(t,f) \right]^* \mathrm{E}\left[e^{i\theta_x(t,f)} \right] \mathrm{E}\left[|T_x(t,f)| \right],
\end{aligned}
$$

which is $O(1/N)$.

- Finally, the ninth term, $\mathrm{E}(S_9)$, yields

$$
\begin{aligned}
\mathrm{E}(S_9) &= \mathrm{E}\left\{\frac{1}{N^4}\sum_{m\neq n} e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|\sum_{k\notin\{l,m,n\}}|T_{x_k}(t,f)|\right\} \\
&= \frac{1}{N^4}\sum_{m\neq n}\mathrm{E}\left\{e^{i[\theta_{x_m}(t,f)-\theta_{x_n}(t,f)]}\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|\sum_{k\notin\{l,m,n\}}|T_{x_k}(t,f)|\right\} \\
&= \frac{1}{N^4}\sum_{m\neq n}\mathrm{E}\left[e^{i\theta_{x_m}(t,f)}\right]\mathrm{E}\left[e^{-i\theta_{x_n}(t,f)}\right]\mathrm{E}\left[\sum_{l\notin\{m,n\}}|T_{x_l}(t,f)|\sum_{k\notin\{l,m,n\}}|T_{x_k}(t,f)|\right] \\
&= \frac{1}{N^4}\sum_{m\neq n}\mathrm{E}\left[e^{i\theta_{x_m}(t,f)}\right]\mathrm{E}\left[e^{i\theta_{x_n}(t,f)}\right]^*\sum_{l\notin\{m,n\}}\mathrm{E}\left[|T_{x_l}(t,f)|\sum_{k\notin\{l,m,n\}}|T_{x_k}(t,f)|\right] \\
&= \frac{1}{N^4}\sum_{m\neq n}\mathrm{E}\left[e^{i\theta_{x_m}(t,f)}\right]\mathrm{E}\left[e^{i\theta_{x_n}(t,f)}\right]^*\sum_{l\notin\{m,n\}}\mathrm{E}\left[|T_{x_l}(t,f)|\right]\mathrm{E}\left[\sum_{k\notin\{l,m,n\}}|T_{x_k}(t,f)|\right] \\
&= \frac{1}{N^4}\sum_{m\neq n}\mathrm{E}\left[e^{i\theta_{x_m}(t,f)}\right]\mathrm{E}\left[e^{i\theta_{x_n}(t,f)}\right]^*\sum_{l\notin\{m,n\}}\mathrm{E}\left[|T_{x_l}(t,f)|\right]\sum_{k\notin\{l,m,n\}}\mathrm{E}\left[|T_{x_k}(t,f)|\right] \\
&= \frac{1}{N^4}\sum_{m\neq n}\mathrm{E}\left[e^{i\theta_{x}(t,f)}\right]\mathrm{E}\left[e^{i\theta_{x}(t,f)}\right]^*\sum_{l\notin\{m,n\}}\mathrm{E}\left[|T_{x}(t,f)|\right]\sum_{k\notin\{l,m,n\}}\mathrm{E}\left[|T_{x}(t,f)|\right] \\
&= \frac{1}{N^4}\sum_{m\neq n}\left|\mathrm{E}\left[e^{i\theta_{x}(t,f)}\right]\right|^2\sum_{l\notin\{m,n\}}\mathrm{E}\left[|T_{x}(t,f)|\right]\sum_{k\notin\{l,m,n\}}\mathrm{E}\left[|T_{x}(t,f)|\right] \\
&= \frac{N(N-1)(N-2)(N-3)}{N^4}\left|\mathrm{E}\left[e^{i\theta_{x}(t,f)}\right]\right|^2\mathrm{E}\left[|T_{x}(t,f)|\right]^2 \\
&= \frac{(N-1)(N-2)(N-3)}{N^3}\left|\mathrm{E}\left[e^{i\theta_{x}(t,f)}\right]\right|^2\mathrm{E}\left[|T_{x}(t,f)|\right]^2.
\end{aligned}
$$

This is the only term that is $O(1)$.

Putting all calculations together, we obtain the following

$$
\begin{aligned}
\mathrm{E}(\mathrm{avgAMP}^2 \times \mathrm{ITC}^2) &= \mathrm{E}\left[\sum_{i=1}^{9} S_i\right] \\
&= \sum_{i=1}^{9} \mathrm{E}[S_i] \\
&= \frac{(N-1)(N-2)(N-3)}{N^3} \left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right|^2 \mathrm{E}\left[|T_x(t,f)|\right]^2 + O\left(\frac{1}{N}\right) \\
&= \left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\left(1-\frac{3}{N}\right) \left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right|^2 \mathrm{E}\left[|T_x(t,f)|\right]^2 + O\left(\frac{1}{N}\right) \\
&= \left|\mathrm{E}\left[e^{i\theta_x(t,f)}\right]\right|^2 \mathrm{E}\left[|T_x(t,f)|\right]^2 + O\left(\frac{1}{N}\right).
\end{aligned}
$$