Online supplement for manuscript "Exponential decay of pairwise correlation in Gaussian graphical models with an equicorrelational one-dimensional connection pattern"

Contents

1 Results for $\boldsymbol{Y}^{(n)}$

1.1 Bounds

We start from Equation (4) of the manuscript. From this equation, it is obvious that $\Psi_{ij}^{(n)} > 0$. Since $u \mapsto 1 - e^{-2u\lambda}$ is a strictly increasing function of u and $u \mapsto 1 - e^{-2[(n+1-u)]\lambda}$ a strictly decreasing function of u, the term in the square root is smaller than one for $i < j$ and

$$\Psi_{ij}^{(n)} < e^{-(j-i)\lambda}.$$

We therefore still have that pairwise correlation decreases exponentially with distance.

1.2 Limit

We can now provide the limit of $\Psi_{ij}^{(n)}$ when $n \to \infty$. Still from Equation (4) of the manuscript, we have

$$\Psi_{ij}^{(n)} \overset{n\to\infty}{\to} e^{-(j-i)\lambda} \sqrt{\frac{1-e^{-2i\lambda}}{1-e^{-2j\lambda}}} \equiv \Psi_{ij}^{(\infty)}. \tag{1}$$

Note that, in this case, $\Psi_{ij}^{(\infty)}$ is a function of both i and j that cannot be expressed as a function of $j-i$ (the distance between i and j) only. An upper bound for $\Psi_{ij}^{(\infty)}$ is given by

$$\Psi_{ij}^{(\infty)} < e^{-(j-i)\lambda}.$$

Also, still from Equation (4) of the manuscript, we have

$$\Psi_{ij}^{(n)} = \Psi_{ij}^{(\infty)} \sqrt{\frac{1-e^{-2(n+1-j)\lambda}}{1-e^{-2(n+1-i)\lambda}}}. \tag{2}$$

Since $u \mapsto 1-e^{-2(n+1-u)\lambda}$ is a strictly decreasing function of u, we obtain for $i<j$

$$\sqrt{\frac{1-e^{-2(n+1-j)\lambda}}{1-e^{-2(n+1-i)\lambda}}} < 1,$$

so that

$$\Psi_{ij}^{(n)} < \Psi_{ij}^{(\infty)}$$

for all n.

1.3 Asymptotic behavior

Using the Taylor expansion of Equation (7) of the manuscript, we can express $\Psi_{ij}^{(n)}/\Psi_{ij}^{(\infty)}$ as

$$\begin{aligned}
\frac{\Psi_{ij}^{(n)}}{\Psi_{ij}^{(\infty)}} &= \left[1-e^{-2(n+1-j)\lambda}\right]^{\frac{1}{2}} \left[1-e^{-2(n+1-i)\lambda}\right]^{-\frac{1}{2}} \\
&= \left\{1-\frac{1}{2}e^{-2(n+1-j)\lambda} + o\left[e^{-2(n+1)\lambda}\right]\right\} \\
&\qquad \times \left\{1+\frac{1}{2}e^{-2(n+1-i)\lambda} + o\left[e^{-2(n+1)\lambda}\right]\right\} \\
&= 1-\frac{e^{2j\lambda}-e^{2i\lambda}}{2}e^{-2(n+1)\lambda} + o\left[e^{-2(n+1)\lambda}\right].
\end{aligned} \tag{3}$$

This result directly entails that

$$\Psi_{ij}^{(n)} - \Psi_{ij}^{(\infty)} = \Psi_{ij}^{(\infty)} \left[\frac{\Psi_{ij}^{(n)}}{\Psi_{ij}^{(\infty)}} - 1\right] = O\left[e^{-2(n+1)\lambda}\right].$$

1.4 General results

All previous results were proved for $i < j$. The case $j < i$ is solved by using the symmetry identity $\Psi_{ij}^{(n)} = \Psi_{ji}^{(n)}$, so that i, j and $j - i$ are replaced with $\min(i, j)$, $\max(i, j)$ and $|j - i|$, respectively. In the end, setting

$$\Psi_{ij}^{(\infty)} = e^{-|j-i|\lambda} \sqrt{\frac{1 - e^{-2\min(i,j)\lambda}}{1 - e^{-2\max(i,j)\lambda}}}, \tag{4}$$

we obtain the following results:

- Bounds: $0 < \Psi_{ij}^{(n)} < \Psi_{ij}^{(\infty)} < e^{-|j-i|\lambda}$;
- Limit: $\Psi_{ij}^{(n)} \to \Psi_{ij}^{(\infty)}$ as $n \to \infty$;
- Asymptotic expansion:

$$\frac{\Psi_{ij}^{(n)}}{\Psi_{ij}^{(\infty)}} = 1 - \frac{1}{2}\left[e^{2\max(i,j)\lambda} - e^{2\min(i,j)\lambda}\right] e^{-2(n+1)\lambda} + o\left[e^{-2(n+1)\lambda}\right]$$

and

$$\Psi_{ij}^{(n)} - \Psi_{ij}^{(\infty)} = O\left[e^{-2(n+1)\lambda}\right].$$

2 Circulant partial correlation matrix

2.1 Model

Assume that $\boldsymbol{X}^{(n)}$ is a Gaussian graphical model with a conditional independence graph given by Figure 3 of the manuscript. The corresponding partial correlation matrix is then given by the following n-by-n symmetric circulant matrix

$$\Pi_{ij}^{(n)} = \begin{cases} 1 & \text{if } i = j \\ \tau & \text{if } |j - i| \in \{1, n - 1\} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

It can be expressed in the general form of circulant matrices as

$$\boldsymbol{\Pi}^{(n)} = \text{circ}\left[c_0^{(n)}, c_1^{(n)}, \cdots, c_{n-1}^{(n)}\right]$$

with $c_0^{(n)} = 1$, $c_1^{(n)} = c_{n-1}^{(n)} = \tau$ and 0 otherwise.

2.2 Pairwise correlation

In this case, $\mathbf{\Upsilon}^{(n)} = 2\boldsymbol{I}_n - \mathbf{\Pi}^{(n)}$ is a symmetric circulant matrix as well with

$$\mathbf{\Upsilon}^{(n)} = \text{circ}(1, -\tau, 0, \ldots, 0, -\tau).$$

For $n \geq 2$, the n eigenvalues of $\mathbf{\Upsilon}^{(n)}$ are given by (Chen, 1987; Chao, 1988)

$$\mu_k^{(n)} = 1 - 2\tau\cos(k\theta_n), \qquad k = 0, \ldots, n-1, \tag{6}$$

where we set $\theta_n = 2\pi/n$. Note that we have $\mu_0^{(n)} = 1 - 2\tau$; for $k \geq 1$, $\mu_{n-k}^{(n)} = \mu_k^{(n)}$; for n even, we also have $\mu_{\frac{n}{2}}^{(n)} = 1 + 2\tau$. Let $\boldsymbol{Q}^{(n)} = n(\mathbf{\Upsilon}^{(n)})^{-1}$, so that $\mathbf{\Sigma}^{(n)} = (\mathbf{\Upsilon}^{(n)})^{-1} = \frac{1}{n}\boldsymbol{Q}^{(n)}$. Then $\boldsymbol{Q}^{(n)}$ is also a symmetric circulant matrix,

$$\boldsymbol{Q}^{(n)} = \text{circ}\left[q_0^{(n)}, q_1^{(n)}, \ldots, q_{n-1}^{(n)}\right],$$

with (Chao, 1988)

$$q_k^{(n)} = \sum_{j=0}^{n-1} \frac{e^{-ijk\theta_n}}{\mu_j^{(n)}} = \sum_{j=0}^{n-1} \frac{e^{-ijk\theta_n}}{1 - 2\tau\cos(j\theta_n)}. \tag{7}$$

In particular, we have for the diagonal term ($k = 0$)

$$q_0^{(n)} = \sum_{j=0}^{n-1} \frac{1}{\mu_j^{(n)}} = \sum_{j=0}^{n-1} \frac{1}{1 - 2\tau\cos(j\theta_n)}. \tag{8}$$

Since $\boldsymbol{Q}^{(n)}$ is a symmetric circulant matrix, so is $\mathbf{\Sigma}^{(n)}$,

$$\mathbf{\Sigma}^{(n)} = \text{circ}\left[\sigma_0^{(n)}, \sigma_1^{(n)}, \ldots, \sigma_{n-1}^{(n)}\right],$$

with

$$\sigma_k^{(n)} = \frac{q_k^{(n)}}{n}. \tag{9}$$

Finally, the correlation matrix $\mathbf{\Omega}^{(n)}$ is also a symmetric circulant matrix,

$$\mathbf{\Omega} = (\Omega_{ij}^{(n)}) = \text{circ}\left[1, \omega_1^{(n)}, \ldots, \omega_{n-1}^{(n)}\right]$$

with

$$\omega_k^{(n)} = \frac{\sigma_k^{(n)}}{\sqrt{[\sigma_0^{(n)}]^2}} = \frac{\sigma_k^{(n)}}{\sigma_0^{(n)}} = \frac{q_k^{(n)}}{q_0^{(n)}}, \tag{10}$$

with $q_k^{(n)}$ and $q_0^{(n)}$ given by Equations (7) and (8), respectively.

2.3 Riemannian sum

Let h_k be the function that maps any $x \in [0, 2\pi]$ to

$$h_k(x) = \frac{e^{-ikx}}{1 - 2\tau\cos(x)}. \tag{11}$$

Setting $x_j^{(n)} = j\theta_n$ for $j = 0, \ldots, n$, we have

$$0 = x_0^{(n)} < x_1^{(n)} < \cdots < x_n^{(n)} = 2\pi.$$

We now define

$$S_k^{(n)} = \sum_{j=0}^{n-1} h_k[x_j^{(n)}][x_{j+1}^{(n)} - x_j^{(n)}] = \theta_n \sum_{j=0}^{n-1} \frac{e^{-ijk\theta_n}}{1 - 2\tau\cos(j\theta_n)} = \theta_n q_k^{(n)}. \tag{12}$$

By construction, $S_k^{(n)}$ is a left Riemann sum that converges to

$$S_k^{(n)} \overset{n\to\infty}{\to} I_k = \int_0^{2\pi} h_k(x)\,\mathrm{d}x.$$

2.4 Computation of integral

We therefore need to compute I_k. Using Euler's formula

$$e^{ix} = \cos(x) + i\sin(x),$$

we obtain

$$I_k = \int_0^{2\pi} \frac{e^{-ikx}}{1 - \tau\left(e^{ix} + e^{-ix}\right)}\,\mathrm{d}x.$$

Performing the parameter change $z = e^{ix}$, we can now write this integral as a contour integral on the unit circle

$$\begin{aligned} I_k &= \oint_{|z|=1} \frac{\overline{z}^k}{1 - \tau(z + z^{-1})} \frac{\mathrm{d}z}{iz} \\ &= \frac{1}{i} \oint_{|z|=1} \frac{\overline{z}^k}{-\tau z^2 + z - \tau}\,\mathrm{d}z. \end{aligned}$$

The roots of $-\tau z^2 + z - \tau$ are given by

$$\alpha = \frac{1 - \sqrt{1 - 4\tau^2}}{2\tau}$$

and $1/\alpha$. The integral therefore yields

$$I_k = -\frac{1}{i\tau} \oint_{|z|=1} \frac{\overline{z}^k}{(z - \alpha)\left(z - \frac{1}{\alpha}\right)}\,\mathrm{d}z.$$

Factoring the fraction yields

$$\frac{1}{(z-\alpha)\left(z-\frac{1}{\alpha}\right)} = \frac{u}{z-\alpha} - \frac{u}{z-\frac{1}{\alpha}}$$

with

$$u = \frac{\alpha}{\alpha^2-1} = -\frac{\tau}{\sqrt{1-4\tau^2}}.$$

We therefore have for the integral

$$I_k = \frac{1}{i\sqrt{1-4\tau^2}} \oint_{|z|=1} \left(\frac{\overline{z}^k}{z-\alpha} - \frac{\overline{z}^k}{z-\frac{1}{\alpha}} \right) \mathrm{d}z.$$

$1/\alpha$ is outside the unit circle, so that

$$\oint_{|z|=1} \frac{\overline{z}^k}{z-\frac{1}{\alpha}} \mathrm{d}z = 0.$$

For the other other integral, we need to compute the residual of $f(z) = \overline{z}^k/(z-\alpha)$ at $z=\alpha$. Since it is a simple pole, we have

$$\mathrm{Res}_{z=\alpha} f(z) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\overline{z}^k = \alpha^k,$$

since $\alpha \in \mathbb{R}$. We are then then led to

$$\oint_{|z|=1} \frac{\overline{z}^k}{z-\alpha} \mathrm{d}z = 2i\pi\alpha^k$$

and, finally,

$$I_k = \frac{2\pi\alpha^k}{\sqrt{1-4\tau^2}}. \tag{13}$$

In particular, we have for $k=0$

$$I_0 = \frac{2\pi}{\sqrt{1-4\tau^2}}. \tag{14}$$

2.5 Asymptotic approximation

Now that we computed I_k, we can go back to the pairwise correlation. Since the sum $S_k^{(n)}$ of Equation (12) converges toward the integral I_k, we have for $\sigma_k^{(n)}$, using Equations (9) and (12),

$$\sigma_k^{(n)} = \frac{q_k^{(n)}}{n} = \frac{S_k^{(n)}}{n\theta_n} = \frac{S_k^{(n)}}{2\pi} \overset{n\to\infty}{\to} \frac{I_k}{2\pi} = \frac{\alpha^k}{\sqrt{1-4\tau^2}} \tag{15}$$

and for $\omega_k^{(n)}$, using Equations (10) and (12),

$$\omega_k^{(n)} = \frac{q_k^{(n)}}{q_0^{(n)}} = \frac{S_k^{(n)}}{S_0^{(n)}} \overset{n\to\infty}{\to} \frac{I_k}{I_0} = \alpha^k. \tag{16}$$

Since $k = |j-i|$, we can conclude that

$$\Omega_{ij}^{(n)} \overset{n\to\infty}{\to} \alpha^{|j-i|}.$$

References

Chao, C.-Y., 1988. A remark on symmetric circulant matrices. Linear Algebra and its Applications 103, 133–148.

Chen, M., 1987. On the solution of circulant lilnear systems. SIAM Journal on Numerical Analysis 24 (3), 668–683.