Supplementary material for manuscript "Time-frequency analysis of event-related brain recordings: Effect of noise on power" by Marrelec et al.

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1 Some basic results

If K random variables X_1, \ldots, X_K are independent, then the expectation of their product is equal to the product of their expectations (Anderson, 1958, §2.2.3)

$$\operatorname{E}\left(\prod_{k=1}^{K} X_{k}\right) = \prod_{k=1}^{K} \operatorname{E}\left(X_{k}\right).$$
(S-1)

A circular random variable is a random variable that, like an angle, is defined on the unit circle, i.e., whose value is only relevant modulo 2π . A key quantity for any circular random variable θ is its circular mean, defined as

$$\mathbf{E}\left(e^{i\theta}\right) = \int \mathbf{p}(\theta) \, e^{i\theta} \, \mathrm{d}\theta. \tag{S-2}$$

The argument of $E(e^{i\theta})$ is the mean angle or mean direction, while the modulus of $E(e^{i\theta})$ is the mean resultant length. It lays between 0 and 1 and is a measure of concentration of $p(\theta)$.

A circular variable θ is said to follow a von Mises distribution with mean direction θ_0 and concentration parameter κ , denoted $\theta \sim \text{VonMises}(\theta_0, \kappa)$, if its distribution is given by (Mardia and Jupp, 2000, §3.5.4)

$$\mathbf{p}(\theta) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(\theta - \theta_0)},$$

where $I_0(\kappa)$ is the modified Bessel function of order 0,

$$I_0(\kappa) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\kappa \cos(\theta - \theta_0)} \,\mathrm{d}\theta.$$
 (S-3)

The usual Gaussian distribution with mean μ and variance σ^2 is denoted by $\mathcal{N}(\mu, \sigma^2)$.

Finally, a function f(N) is said to be O(1/N) if it is bounded by a function proportional to 1/N, that is, for which there exists an N_0 and a k > 0 such that

$$\forall N > N_0 \quad |f(N)| < \frac{k}{N}.$$

2 S-transform

The S-transform of a signal s(t) is defined as (Stockwell et al., 1996)

$$T_s(t,f) = \frac{|f|}{\sqrt{2\pi}} \int s(u) \, e^{-\frac{1}{2}f^2(u-t)^2} e^{-2i\pi f u} \, \mathrm{d}u.$$
(S-4)

We can express it as

$$T_s(t,f) = \int s(u)\phi_{t,f}(u)^* \,\mathrm{d}u \tag{S-5}$$

with

$$\phi_{t,f}(u) = \frac{|f|}{\sqrt{2\pi}} e^{-\frac{1}{2}f^2(u-t)^2} e^{2i\pi f u}.$$
(S-6)

The Fourier transform of $\phi_{t,f}$ is given by

$$\widehat{\phi_{t,f}}(\nu) = \frac{|f|}{\sqrt{2\pi}} \int e^{-\frac{1}{2}f^2(u-t)^2} e^{2i\pi(f-\nu)u} \,\mathrm{d}u.$$
(S-7)

This quantity can be understood as the value of the characteristic function of a normal distribution with mean t and variance $1/f^2$ calculated at point $2\pi(f-\nu)$, which is equal to (Johnson et al., 1994, Eq. (13.13))

$$\widehat{\phi_{t,f}}(\nu) = e^{-\frac{1}{2}(2\pi)^2 \left(1 - \frac{\nu}{f}\right)^2} e^{-2i\pi t(\nu - f)}.$$
(S-8)

The module of this quantity is given by $e^{-\frac{1}{2}(2\pi)^2 \left(1-\frac{\nu}{f}\right)^2}$. For f > 0, we have

$$|\widehat{\phi_{t,f}}(\nu)| \le e^{-\frac{1}{2}(2\pi)^2} \approx 2.73 \times 10^{-9}, \qquad \nu < 0.$$

Since this is very small, $\phi_{t,f}(u)$ is approximately an analytic function.

Consider now f > 0. Since s(u) is real, it is easy to show that

$$T_s(t, -f) = T_s(t, f)^*.$$
 (S-9)

Since $\phi_{t,f}(u)$ is approximately an analytic function and the signals we deal with are real, a common approach is to apply the S-transform to $s_a(u)$, the analytic signal associated to s(u), defined as the signal whose Fourier transform is related to the original one by (Mallat, 1999, §4.3.2)

$$\widehat{s_a}(\nu) = \begin{cases} 2\widehat{s}(\nu) & \text{for } \nu \ge 0\\ 0 & \text{for } \nu < 0, \end{cases}$$
(S-10)

where $\hat{s}(\nu)$ is the Fourier transform of s(u). Applying Plancherel theorem, we obtain for $T_{s_a}(t, f)$

$$\int s_a(u)\phi_{t,f}(u)^* du = \int \widehat{s_a}(\nu)\widehat{\phi_{t,f}}(\nu)^* d\nu$$

$$= \underbrace{\int_{-\infty}^0 \widehat{s_a}(\nu)\widehat{\phi_{t,f}}(\nu)^* d\nu}_{=0} + \int_{0}^{+\infty} \underbrace{\widehat{s_a}(\nu)}_{=2\widehat{s}(\nu)} \widehat{\phi_{t,f}}(\nu)^* d\nu$$

$$= 2\int_{0}^{+\infty} \widehat{s}(\nu)\widehat{\phi_{t,f}}(\nu)^* d\nu$$

$$\approx 2\int \widehat{s}(\nu)\widehat{\phi_{t,f}}(\nu)^* d\nu$$

$$= 2\int s(u)\phi_{t,f}(u)^* du$$

$$= 2T_s(t, f).$$

As a conclusion, we have, for f > 0,

$$T_{s_a}(t,f) \approx 2T_s(t,f). \tag{S-11}$$

3 Time-frequency transform of noise

3.1 Wide-sense stationary noise

Since $E[T_b(t, f)] = 0$, we have

$$\operatorname{Var}\left[T_b(t,f)\right] = \operatorname{E}\left[|T_b(t,f)|^2\right].$$

We express $|T_b(t, f)|^2$ as

$$|T_b(t,f)|^2 = \left[\int b(u)\phi_{t,f}^*(u) \,\mathrm{d}u\right] \left[\int b(v)\phi_{t,f}^*(v) \,\mathrm{d}v\right]^*$$
$$= \int b(u)b(v)\phi_{t,f}^*(u)\phi_{t,f}(v) \,\mathrm{d}u \,\mathrm{d}v.$$

As a consequence, we obtain

$$\mathbf{E}\left[|T_b(t,f)|^2\right] = \int \mathbf{E}\left[b(u)b(v)\right]\phi_{t,f}^*(u)\phi_{t,f}(v)\,\mathrm{d}u\,\mathrm{d}v.$$

For a wide-sense stationary noise (i.e., a noise for which the mean and variance are time independent, and for which the autocorrelation function only depends on the lag between time points), Wiener-Khinchin theorem yields (Bendat and Piersol, 1986, $\S5.2$)

$$E[b(u)b(v)] = R_b(v-u) = \int S_b(v)e^{2i\pi\nu(v-u)}d\nu,$$
 (S-12)

where $S_b(\nu)$ is the power spectral density (PSD) of b(t). We therefore have for $\mathbb{E}\left[|T_b(t,f)|^2\right]$

$$E\left[|T_{b}(t,f)|^{2}\right] = \int \left[\int S_{b}(\nu)e^{2i\pi\nu(\nu-u)}d\nu\right]\phi_{t,f}^{*}(u)\phi_{t,f}^{*}(\nu)dud\nu$$

$$= \int S_{b}(\nu)\left[\int \int \phi_{t,f}^{*}(u)e^{-2i\pi\nu(\nu-u)}dud\nu\right]d\nu$$

$$= \int S_{b}(\nu)\left[\int \phi_{t,f}^{*}(\nu)e^{-2i\pi\nu u}du\right]\left[\int \phi_{t,f}(\nu)e^{2i\pi\nu \nu}d\nu\right]d\nu$$

$$= \int S_{b}(\nu)\left|\widehat{\phi_{t,f}^{*}}(\nu)\right|^{2}d\nu$$

$$= \int S_{b}(\nu)\left|\widehat{\phi_{t,f}}(\nu)\right|^{2}d\nu.$$
(S-13)

3.2 Case of white noise

In the case of a Gaussian white noise, the power spectral density of b(t) can be calculated through the autocorrelation function using the inverse relationship of Equation (S-12) (Bendat and Piersol, 1986, §5.2)

$$S_b(\nu) = \widehat{R_b}(\nu) = \int R_b(u) e^{-2i\pi\nu u} \,\mathrm{d}u. \tag{S-14}$$

The integral can be calculated using the approximation by the corresponding Riemann sum

$$S_b(\nu) \approx \delta t \sum_k R_b(k\delta t) e^{-2i\pi\nu k\delta t},$$
 (S-15)

with

$$R_b(0) = \mathbf{E}\left[b(t)^2\right] = \sigma^2, \tag{S-16}$$

and, for $k \neq 0$,

$$R_b(k\delta t) = \mathbb{E}\left[b(t)b(t+k\delta t)\right] = 0, \qquad (S-17)$$

so that

$$S_b(\nu) = \sigma^2 \delta t \tag{S-18}$$

and

$$\mathbb{E}\left[|T_b(t,f)|^2\right] = \sigma^2 \,\delta t \int \left|\widehat{\phi_{t,f}}(\nu)\right|^2 \mathrm{d}\nu$$

= $\sigma^2 \,\delta t \int |\phi_{t,f}(u)|^2 \,\mathrm{d}u.$ (S-19)

The integral can be computed from the definition of $\phi_{t,f}(u)$ in the case of the S-transform, Equation (4) of the manuscript, as

$$\int |\phi_{t,f}(u)|^2 \, \mathrm{d}u = \frac{2|f|^2}{\pi} \int e^{-f^2(u-t)^2} \, \mathrm{d}u$$
$$= \frac{2|f|}{\sqrt{\pi}},$$

so that

$$\mathbf{E}\left[\left|T_{b}(t,f)\right|^{2}\right] = \frac{2|f|\sigma^{2}\delta t}{\sqrt{\pi}}.$$
(S-20)

3.3 Case of color noise

We assume that b(t) has a PSD such that most of its energy is a domain where $S_b(\nu)$ is of the form

$$S_b(\nu) \propto \frac{1}{\nu^c}.$$
 (S-21)

 $E[|T_b(t, f)|^2]$ can then be calculated from Equation (S-13), yielding

$$\mathbf{E}\left[|T_b(t,f)|^2\right] \propto \int \frac{1}{\nu^c} \left|\widehat{\phi_{t,f}}(\nu)\right|^2 \mathrm{d}\nu.$$

For the S-transform, we obtain

$$E\left[|T_b(t,f)|^2\right] \propto \int \frac{1}{\nu^c} e^{-(2\pi)^2 \left(1 - \frac{\nu}{f}\right)^2} d\nu.$$
 (S-22)

Performing the parameter change $\nu \mapsto x = \nu/f$, we are led to

$$\begin{split} \mathbf{E} \begin{bmatrix} |T_b(t,f)|^2 \end{bmatrix} & \propto \quad \frac{1}{f^{c-1}} \int \frac{1}{x^c} e^{-(2\pi)^2 (1-x)^2} \mathrm{d}x \\ & \propto \quad f^{-(c-1)}. \end{split}$$

4 Morlet wavelet

4.1 Definition

The continuous wavelet transform with Morlet wavelet for a scale a and shift b is defined as

$$\int s(u) \,\psi_{a,b}^*(u) \,\mathrm{d}u \tag{S-23}$$

with

$$\psi_{a,b}(u) = \frac{1}{\sqrt{a}}\psi\left(\frac{u-b}{a}\right) \tag{S-24}$$

and

$$\psi(u) = c_r \pi^{-\frac{1}{4}} e^{-\frac{u^2}{2}} \left(e^{iru} - d_r \right), \qquad (S-25)$$

where r is a parameter, and d_r and c_r are defined by

$$d_r = e^{-\frac{1}{2}r^2}$$
 and $c_r = \left(1 + e^{-r^2} - 2e^{-\frac{3}{4}r^2}\right)^{-\frac{1}{2}}$. (S-26)

 ψ has the following properties: its L_2 -norm is equal to 1 and its integral is equal to 0. For r large enough, d_r is small (e.g., for r > 5, $d_r < 4 \times 10^{-6}$) and can be neglected, leading to the approximate form of the Morlet transform

$$\psi(u) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}u^2} e^{iru}.$$
 (S-27)

For reasons that will become clear below, we set $r = 2\pi q$, so that $\psi(u)$ can be expressed as

$$\psi(u) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}u^2} e^{2i\pi qu},$$
(S-28)

and $\psi_{a,b}(u)$ as

$$\psi_{a,b}(u) = \frac{\pi^{-\frac{1}{4}}}{\sqrt{a}} e^{-\frac{(u-b)^2}{2a^2}} e^{2i\pi q \frac{u-b}{a}}.$$
(S-29)

4.2 Fourier transform

The Fourier transform of $\psi_{a,b}(u)$ is given by

$$\widehat{\psi_{a,b}}(\nu) = \int \psi_{a,b}(u) e^{-2i\pi\nu u} du$$

= $\frac{\pi^{-\frac{1}{4}}}{\sqrt{a}} \int e^{-\frac{(u-b)^2}{2a^2}} e^{2i\pi \left(q\frac{u-b}{a} - \nu u\right)} du$
= $\pi^{\frac{1}{4}} \sqrt{2a} e^{-2i\pi \frac{q}{a}b} \left[\frac{1}{\sqrt{2\pi a^2}} \int e^{-\frac{(u-b)^2}{2a^2}} e^{2i\pi \left(\frac{q}{a} - \nu\right)u} du\right]$

The term between brackets is the characteristic function of a Gaussian distribution with mean b and variance a^2 taken at value $2\pi(\frac{q}{a}-\nu)$, which is equal to (Johnson et al., 1994, Eq. (13.13))

$$e^{2i\pi\left(\frac{q}{a}-\nu\right)b-\frac{1}{2}(2\pi)^2a^2\left(\frac{q}{a}-\nu\right)^2}.$$

In the end, we obtain

$$\widehat{\psi_{a,b}}(\nu) = \pi^{\frac{1}{4}} \sqrt{2a} e^{-\frac{1}{2}(2\pi)^2 a^2 \left(\frac{a}{a} - \nu\right)^2} e^{-2i\pi b\nu}.$$
(S-30)

Its power is given by

$$|\widehat{\psi_{a,b}}(\nu)|^2 = 2a\sqrt{\pi}e^{-(2\pi)^2a^2\left(\frac{q}{a}-\nu\right)^2}.$$
(S-31)

The central frequency ν_{\max} of $\psi_{a,b}(u)$, which is the frequency for which $|\widehat{\psi}_{a,b}(\nu)|^2$ is maximal, is given by $\nu_{\max} = q/a$. This is independent of b. In particular, q is the central frequency of $\psi(u)$.

4.3 Relating location-scale and time-frequency

Equation (S-23) is of the form of Equation (3) of the manuscript with $\phi_{t,f}(u) = \psi_{a,b}(u)$. To specify this relationship, we need to relate (t, f) and (a, b). If we set b = t and a = q/f (i.e., f = q/a), we obtain

$$\phi_{t,f}(u) = \psi_{\frac{q}{f},t}(u) = \pi^{-\frac{1}{4}} \sqrt{\frac{f}{q}} e^{-\frac{1}{2} \left(\frac{f}{q}\right)^2 (u-t)^2} e^{2i\pi f(u-t)},$$
(S-32)

with power

$$|\phi_{t,f}(u)|^2 = \frac{f}{q\sqrt{\pi}} e^{-\left(\frac{f}{q}\right)^2 (u-t)^2}.$$
(S-33)

We see that the maximum of $|\phi_{t,f}(u)|^2$ is reached for u = t. From Equation (S-30), the Fourier transform is given by

$$\widehat{\phi_{t,f}}(\nu) = \pi^{\frac{1}{4}} \sqrt{\frac{2q}{f}} e^{-\frac{1}{2}(2\pi)^2 q^2 \left(1 - \frac{\nu}{f}\right)^2} e^{-2i\pi\nu t},$$
(S-34)

with power equal to

$$|\widehat{\phi_{t,f}}(\nu)|^2 = \frac{2q\sqrt{\pi}}{f} e^{-(2\pi)^2 q^2 \left(1 - \frac{\nu}{f}\right)^2}.$$
(S-35)

The maximum of this quantity is reached for $\nu = f$. The time-frequency transform corresponding to the fonction $\phi_{t,f}(u)$ defined Equation (S-32) then reads

$$T_s(t,f) = \pi^{-\frac{1}{4}} \sqrt{\frac{f}{q}} e^{2i\pi ft} \int s(u) \, e^{-\frac{1}{2} \left(\frac{f}{q}\right)^2 (u-t)^2} e^{-2i\pi fu} \, \mathrm{d}u.$$
(S-36)

4.4 Transform of oscillatory signal

The (approximate) Morlet transform of a complex oscillatory signal of the form $s(u) = \Omega_0 e^{i(2\pi\nu_0 u + \phi_0)}$, i.e., with amplitude Ω_0 , frequency ν_0 and phase ϕ_0 , is given by

$$\Omega_0 \pi^{-\frac{1}{4}} \sqrt{\frac{f}{q}} e^{i(2\pi ft + \phi_0)} \int e^{-\frac{1}{2} \left(\frac{f}{q}\right)^2 (u-t)^2} e^{2i\pi(\nu_0 - f)u} \, \mathrm{d}u$$

$$= \Omega_0 \pi^{\frac{1}{4}} \sqrt{\frac{2q}{f}} e^{i(2\pi ft + \phi_0)} \left[\frac{1}{\sqrt{2\pi \left(\frac{q}{f}\right)^2}} \int e^{-\frac{1}{2} \left(\frac{f}{q}\right)^2 (u-t)^2} e^{2i\pi(\nu_0 - f)u} \, \mathrm{d}u \right].$$
(S-37)

The term in brackets is the characteristic function of a Gaussian distribution with mean t and variance q^2/f^2 calculated at value $2\pi(\nu_0 - f)$, which is equal to (Johnson et al., 1994, Eq. (13.13))

$$e^{2i\pi(\nu_0-f)t}e^{-\frac{1}{2}(2\pi)^2\left(\frac{q}{f}\right)^2(\nu_0-f)^2}$$

In the end, we obtain

$$T_s(t,f) = \Omega_0 \pi^{\frac{1}{4}} \sqrt{\frac{2q}{f}} e^{-\frac{1}{2}(2\pi)^2 q^2 \left(1 - \frac{\nu_0}{f}\right)^2} e^{i(2\pi\nu_0 t + \phi)}.$$
 (S-38)

The power of this quantity is given by

$$|T_s(t,f)|^2 = \Omega_0^2 \frac{2q\sqrt{\pi}}{f} e^{-(2\pi)^2 q^2 \left(1 - \frac{\nu_0}{f}\right)^2},$$
(S-39)

which is maximal for

$$f = 2\pi^2 q^2 \nu_0 \left(\sqrt{1 + \frac{1}{\pi^2 q^2}} - 1 \right) = \alpha \nu_0, \tag{S-40}$$

where we set

$$\alpha = 2\pi^2 q^2 \left(\sqrt{1 + \frac{1}{\pi^2 q^2}} - 1 \right).$$
 (S-41)

For $\pi^2 q^2 \gg 1$, we have $\alpha \approx 1$, i.e., the maximum is reached for f close to ν_0 . The value at $f = \alpha \nu_0$ is given by

$$|T_s(t,\alpha\nu_0)|^2 = \Omega_0^2 \frac{2q\sqrt{\pi}}{\alpha\nu_0} e^{-(2\pi)^2 q^2 \left(1-\frac{1}{\alpha}\right)^2}.$$
 (S-42)

This quantity decreases in $1/\nu_0$.

4.5 White noise

In the case of white noise with variance σ^2 , we have according to Equation (18) of the manuscript

$$\mathbf{E}\left[\left|T_{b}(t,f)\right|^{2}\right] = \sigma^{2} \,\delta t \int \left|\phi_{t,f}(u)\right|^{2} \mathrm{d}u = \sigma^{2} \,\delta t, \qquad (S-43)$$

since the Morlet wavelet is L_2 -normalized to 1. This expectation does not depend on f—compare with Equation (19) of the manuscript.

4.6 Color noise

According to Equation (20) of the manuscript and Equation (S-35), we have

$$\mathbf{E}\left[|T_b(t,f)|^2\right] \propto \frac{1}{f} \int \frac{1}{\nu^c} e^{-(2\pi)^2 q^2 \left(1 - \frac{\nu}{f}\right)^2} \,\mathrm{d}\nu.$$
(S-44)

Performing the parameter change $\nu \mapsto x = \nu/f$, we are led to

$$\mathbf{E}\left[\left|T_{b}(t,f)\right|^{2}\right] \propto f^{-c}.$$
(S-45)

In this case, the variance of the noise time-frequency transform decays as fast as its power spectral density. Again, the result obtained for white noise, Equation (S-43), is compatible with this result with c = 0 (which corresponds to white noise).

4.7 avgPOW and POWavg for pure oscillatory signal

We consider purely oscillatory signals as in $\S3$ of the manuscript. From Equation (1) of the manuscript and Equation (S-39) and , we obtain

$$\operatorname{avgPOW}_{s_{1:N}}(t,f) = \frac{2q\sqrt{\pi}}{f} e^{-(2\pi)^2 q^2 \left(1 - \frac{\nu_0}{f}\right)^2} \frac{1}{N} \sum_{n=1}^N \Omega_n^2,$$
(S-46)

with expectation given by

$$E\left[\operatorname{avgPOW}_{s_{1:N}}(t,f)\right] = \frac{2q\sqrt{\pi}}{f} \left(\Omega_0^2 + \tau_\Omega^2\right) e^{-(2\pi)^2 q^2 \left(1 - \frac{\nu_0}{f}\right)^2}.$$
 (S-47)

Also, from Equations (2) and (7) of the manuscript as well as Equation (S-38), we are led to

$$\text{POWavg}_{s_{1:N}}(t,f) = \frac{2q\sqrt{\pi}}{f} e^{-(2\pi)^2 q^2 \left(1 - \frac{\nu_0}{f}\right)^2} \left| \frac{1}{N} \sum_{n=1}^N \Omega_n e^{i\phi_n} \right|^2,$$
(S-48)

and corresponding expectation

$$E\left[POWavg_{s_{1:N}}(t,f)\right] = \frac{2q\sqrt{\pi}}{f}e^{-(2\pi)^2q^2\left(1-\frac{\nu_0}{f}\right)^2}E\left[\left|\frac{1}{N}\sum_{n=1}^N\Omega_n e^{i\phi_n}\right|^2\right].$$
 (S-49)

We then use the fact that the expectation of the term within brackets can be approximated by (see Appendix C of the manuscript)

$$\mathbf{E}\left[\left|\frac{1}{N}\sum_{n=1}^{N}\Omega_{n}e^{i\phi_{n}}\right|^{2}\right] = \Omega_{0}^{2}\rho^{2} + O\left(\frac{1}{N}\right).$$
(S-50)

Finally,

$$E\left[POWavg_{s_{1:N}}(t,f)\right] = \frac{2q\sqrt{\pi}}{f} \Omega_0^2 \rho^2 e^{-(2\pi)^2 q^2 \left(1 - \frac{\nu_0}{f}\right)^2} + O\left(\frac{1}{N}\right).$$
 (S-51)

4.8 L₁-norm normalization

We here compute the normalization constant for a Morlet wavelet with L_1 normalization, $\phi_{t,f}^{(L_1)}(u)$. From Equation (S-32), we have

$$\phi_{t,f}^{(L_1)}(u) \propto e^{-\frac{1}{2}\left(\frac{f}{q}\right)^2 (u-t)^2} e^{2i\pi f(u-t)},\tag{S-52}$$

In the case of an L_1 -norm normalization, we require

$$\int \left| \phi_{t,f}^{(L_1)}(u) \right| \, \mathrm{d}u = 1. \tag{S-53}$$

The absolute value of the right-hand side of Equation (S-52),

$$e^{-\frac{1}{2}\left(\frac{f}{q}\right)^2(u-t)^2},$$
 (S-54)

is positive and proportional to a Gaussian distribution with mean t and variance $(q/f)^2$. As a consequence,

$$\int e^{-\frac{1}{2}\left(\frac{f}{q}\right)^2 (u-t)^2} \,\mathrm{d}u = \sqrt{2\pi \left(\frac{q}{f}\right)^2} = \frac{q\sqrt{2\pi}}{|f|}.$$
(S-55)

From there, we obtain

$$\phi_{t,f}^{(L_1)}(u) = \frac{|f|}{q\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{f}{q}\right)^2 (u-t)^2} e^{2i\pi f(u-t)}.$$
(S-56)

Comparing this result with Equation (S-6), we see that the difference with the S-transform is twofold:

- In the amplitude of $\phi_{t,f}(u)$, the frequency f (in the case of the S-transform) is changed to f/q (in the case of the Morlet wavelet);
- The phase, which is equal to $2\pi f u$ for the S-transform, is shifted so that it is equal to 0 for u = t.

References

- T. W. Anderson. An Introduction to Multivariate Statistical Analysis. Wiley Publications in Statistics. John Wiley and Sons, New York, 1958.
- J. S. Bendat and A. G. Piersol. *Random Data. Analysis and Measurement Procedures*. John Wiley & Sons, New York, 2nd edition, 1986.
- N. L. Johnson, S. Kotz, and N. Balakrishnan. Continuous Univariate Distributions, volume 1 of Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics Section. John Wiley and Sons, New York, 2nd edition, 1994.
- S. Mallat. A Wavelet Tour of Signal Processing. Wavelet Analysis & Its Applications. Academic Press, Amsterdam, 2nd edition, 1999.
- K. V. Mardia and P. E. Jupp. *Directional Statistics*. Wiley Series in Probability and Statistics. Wiley, Chichester, 2000.
- R. G. Stockwell, L. Mansinha, and R. P. Lowe. Localization of the complex spectrum: the S transform. *IEEE Transactions on Signal Processing*, 44(4):998–1001, 1996.